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A CHARACTERIZATION OF SOME GEOMETRIES OF EXCEPTIONAL LIE TYPE

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A characterization of some geometries of exceptional Lie type *)

by

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ABSTRACT

For geometries associated with permutation representations of the groups of Lie type E_6, E_7, E_8 on certain maximal parabolic subgroups, axiom systems are given that characterize them in terms of points and lines.

KEY WORDS & PHRASES: *geometries of Lie type, buildings, polar spaces, parapolar spaces*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. A graph Γ is always meant to be without loops and without multiple edges. Often, we shall abuse terminology and refer to Γ as the vertex set of the graph Γ. Thus $x \in \Gamma$ means that x is a vertex of Γ. Moreover $\Gamma(x)$ denotes the set of vertices of Γ adjacent to x. For x, $y \in \Gamma$, write $x^{\perp_\Gamma}$, or just $x^\perp$ if Γ is clear from the context, for the set $\{x\} \cup \Gamma(x)$, and write $x \perp_\Gamma y$ (or just $x \perp y$) to denote $y \in x^{\perp_\Gamma}$. The tuple $(P,\perp)$ of a set P and a binary symmetric and reflexive relation $\perp$ will be called *a looped graph*. $(\Gamma,\perp_\Gamma)$ is the *looped graph of* Γ. Any looped graph is of course, the looped graph of a uniquely determined graph. For $x,y \in \Gamma$, denote by $d_\Gamma(x,y)$ (or just $d(x,y)$ whenever no confusion arises) the ordinary distance in Γ.

For X a subset of Γ, put $X^\perp = \cap_{x\in X} x^\perp$. Moreover, if $y \in P$, let $d(y,X) = \inf_{x\in X} d(y,x)$. Instead of $y \in X^\perp$, we shall also write $y \perp X$. An *incidence system* (P,L) is a set P of *points* and a collection L of subsets of P of cardinality at least two, called *lines*. If (P,L) is an incidence system, then the *point graph* or *collinearity graph* of (P,L) is the graph whose looped graph is $(P,\perp)$, where $\perp$ denotes collinearity in (P,L). The incidence system is called *connected* whenever its collinearity graph is connected. Likewise terms such as (co-) cliques, paths will be applied freely to (P,L) when in fact they are meant for its collinearity graph.

A subset X of P is called a *subspace* of (P,L) whenever each point of P on a line bearing two distinct points of X is itself in X. A subspace X is called *singular* whenever it induces a clique in (P,L). The length i of a longest chain $X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_i = X$ of nonempty singular subspaces X_i of X is called the *rank* of the singular subspace X and denoted by rk (X). The *singular rank* of (P,L) is the maximal number s (possibly ∞) for which a singular subspace of (P,L) of rank s can be found. If this number is finite, then (P,L) is said to be of finite singular rank. For a subset X of P, the subspace generated by X is written $\langle X\rangle$. Instead of $\langle X\rangle$ we also write $\langle x,Y\rangle$ if $X = \{x\} \cup Y$, and so on. If F is a family of subsets of P and X is a subset of P, then $F(X)$ denotes the family of members of F contained in X, while F_X stands for the family of members of F containing X. If $X = \{x\}$ for some $x \in P$, we often write F_x instead of $F_{\{x\}}$. Furthermore, if H is another family of subsets of P, then we set

$$F(H) = \{F(\mathrm{H}) | \mathrm{H} \in H\} \quad \text{and} \quad F_{\mathrm{H}} = \{F_{\mathrm{H}} | \mathrm{H} \in H\}.$$

The incidence system (P,L) is called *linear* if any two distinct points are on at most one line. In this case, for a pair x,y of collinear points, xy represents this line; thus $xy = \langle x,y \rangle$. A line is called *thick* if there are at least three points on it, otherwise it is called thin. A path $x_1, x_2, \ldots, x_d$ of points (i.e., $x_i \in x_{i+1}^{\perp}$ for $i=1,..,d-1$) will be called a *geodesic* whenever $d(x_1,x_d) = d$. A subspace X of P will be called *geodesically closed*, if the points of any geodesic whose endpoints belong to X are all contained in X. If the incidence system (P,L) satisfies $P^{\perp} = \emptyset$, it is called *nondegenerate*. Recall from [4] that (P,L) is a *polar space* if $|x^{\perp} \cap L| \neq 1$ implies $L \subseteq x^{\perp}$ for any $x \in P$ and $L \in L$. Polar spaces are linear incidence systems, and maximal singular subspaces exist within polar spaces. The *rank* of a polar space (P,L) is the number k such that k-1 is the singular rank of (P,L). A *generalized quadrangle* is a polar space of rank 2.

1.2. We shall now discuss the axioms for incidence systems (P,L) with which we shall be concerned.

(F1) If $x \in P$ and $L \in L$ with $|x^{\perp} \cap L| > 1$, then $x \perp L$.

This means that (P,L) is a *Gamma space* (in D.G. Higman's terminology). Note (F1) implies that $X^{\perp}$ is a subspace for any subset X of P.

LEMMA 1 (see [9]). *Let (P,L) be a Gamma space. Then*
(i) *For any clique X of P, the subspace $\langle X \rangle$ is singular.*
(ii) *Maximal cliques of P are maximal singular subspaces.*

Any singular subspace of a Gamma space is contained in a maximal singular subspace. The collection of all maximal singular subspaces of (P,L) will be denoted by M.
Here are two more axioms:

(F2) The graph induced on $\{x,y\}^{\perp}$ is not a clique whenever $x \in P$ and $y \in x^{\perp}$.

(F3) If $x,y \in P$ with $d(x,y) = 2$ and $|\{x,y\}^{\perp}| > 1$, then $\{x,y\}^{\perp}$ is a nondegenerate polar space of rank at least 2.

An incidence system satisfying (F1), (F2), (F3) will be called a *parapolar* *space* if it is connected and all its lines are thick. A pair of points x,y of P with $d(x,y) = 2$ is called *symplectic* if $|\{x,y\}^{\perp}| \geq 2$ and *special* otherwise. If x,y is a symplectic pair, there exists a unique geodesically closed subspace $S(x,y)$ of P which is isomorphic to a polar space (cf [2], [9]) as we shall see in Proposition 1 below. This explains the importance of symplectic pairs in parapolar spaces. Their existence is guaranteed by axiom (F3).

The following three axioms are special instances of (F3). Let I be a set of natural numbers ≥ 2.

$(F3)_I$ If $x,y \in P$ with $d(x,y) = 2$, then $\{x,y\}^{\perp}$ is either a single point or a nondegenerate polar space of rank a member of I.

(P3) If $x,y \in P$ with $d(x,y) = 2$, then $\{x,y\}^{\perp}$ is a nondegenerate polar space of rank at least 2.

$(P3)_I$ If $x,y \in P$ with $d(x,y) = 2$, then $\{x,y\}^{\perp}$ is a nondegenerate polar space of rank a member of I.

Note that $(P3)_I$ is stronger than any of (P3) and $(F3)_I$. If $I = \{k\}$, we shall write $(P3)_k$ rather than $(P3)_{\{k\}}$.

For the characterizations we have in mind, we need two more axioms for (P,L).

(F4) If x,y is a symplectic pair in P and L is a line on y with $x^{\perp} \cap L = \emptyset$, then $x^{\perp} \cap L^{\perp}$ is either a point or a maximal clique in $\{x,y\}^{\perp}$.

(P4) If x,y is a symplectic pair in P and L is a line on y with $x^{\perp} \cap L = \emptyset$, then $x^{\perp} \cap L^{\perp}$ is either empty or a maximal clique in $\{x,y\}^{\perp}$.

We now describe how the incidence systems to be characterized are obtained from buildings.

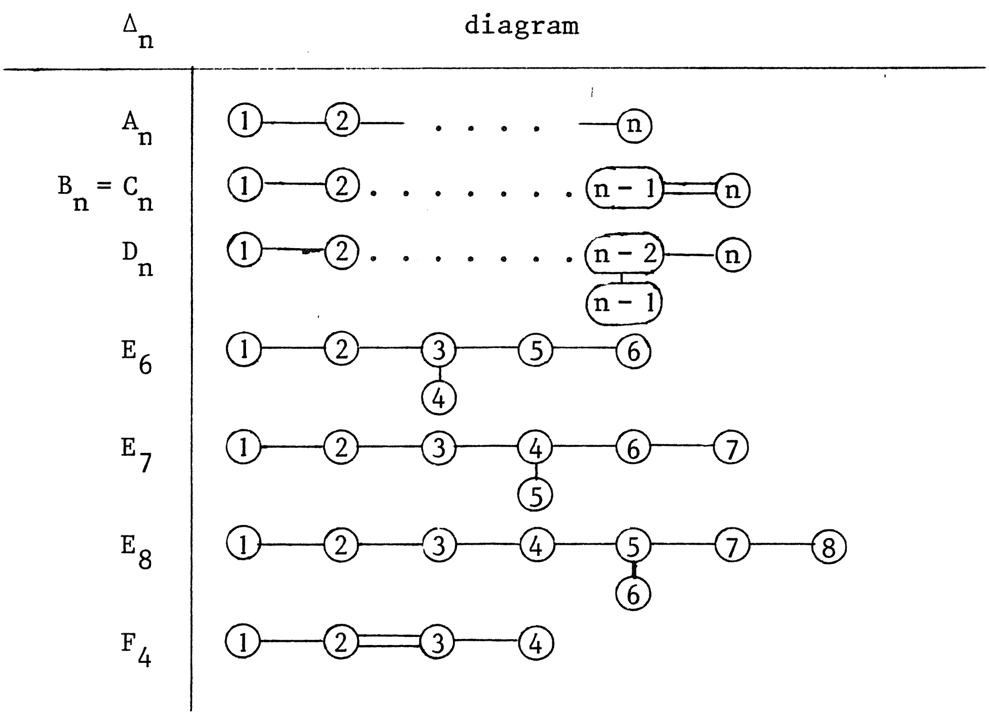
Let Δ_n denote a *Coxeter diagram of spherical type* (see [12]) labelled as in Table 1. The subscript n is the number of nodes of the diagram and often referred to as the *rank* of the diagram. Set $I_n = \{1,2,\ldots,n\}$. It is the set of labels of the nodes of Δ_n.

1.3 We reword the notion of geometry of type Δ_n from Tits [13]. First, a *geometry over* (an index set) I of cardinality n is an n-partite looped graph

$(\Gamma, *)$ with parts Γ_i for $i \in I$ (some of which may be empty). The map $\tau\colon \Gamma \to I$ determined by $x \in \Gamma_{\tau(x)}$ for $x \in \Gamma$, is called the *type map* of Γ and $\tau(X)$ for X an element or subset of Γ, the *type* of X. The number n is called the *rank* of Γ. A *flag* of a geometry Γ over I is a clique of Γ. Two flags are said to be *incident* if their union is a flag. The *rank (corank)* of a flag X is $|X|$ ($n-|X|$ resp.)

Table 1

COXETER DIAGRAMS OF SPHERICAL TYPE WITH RANK $n \geq 3$.



For a geometry Γ we shall often write $*$ rather than $\perp_\Gamma$. Furthermore, we shall refer to two elements γ, δ of Γ as being incident rather than as being adjacent or equal whenever $\gamma * \delta$ hols. (This implies that the notions of incidence for γ, δ and for $\{\gamma\}, \{\delta\}$ are equivalent). Let X be a flag of Γ. Then the subgraph of Γ induced on $Y = X^* \backslash X$ considered as a geometry over $I \backslash \tau(X)$, is called the *residue of* X *in* Γ and denoted by Γ_X.

A geometry over I is called *connected* when Γ is connected and nonempty and it is called *residually connected* if the residue of every flag of corank ≥ 2 is connected and the residue of every flag of corank 1 is nonempty. In a residually connected geometry Γ over I of rank n any flag is contained in a maximal flag (of rank n) and Γ is nonempty for each $i \in I$.

Let m be a natural number ≥ 2. A geometry Γ of rank 2 is called a *generalized* m-*gon* if Γ has diameter 2 and girth 2m and if every vertex of Γ is in at least two edges. If (P,L) is a projective plane, then putting $\Gamma_1 = P$ and $\Gamma_2 = L$ and letting adjacency in $\Gamma = \Gamma_1 \cup \Gamma_2$ stand for incidence in (P,L), we obtain a generalized 3-gon. Similary, a nondegenerate generalized quadrangle leads to a generalized 4-gon. The converse construction is also possible.

Consider a Coxeter diagram Δ_n. For any two labels $i,j \in I_n$ let $m(i,j)$ be the label of the bond between the nodes labelled i and j. Thus $m(i,j) = 2$ if i,j are not adjacent, $m(i,j) = 3$ if they are joined by a single bond, $m(i,j) = 4$ if they are joined by a double bond, and $m(i,j) = 6$ when joined by a triple bond.

A *geometry of type* Δ_n is defined to be a residually connected geometry over I_n such that for any two distinct $i,j \in I_n$ the residue of any flag of type $I_n \setminus \{i,j\}$ is a generalized $m(i,j)$-gon. If X is a flag of Γ and Δ^1 is the subdiagram of Δ_n whose nodes are the members of $I_n \setminus \tau(X)$, then Γ_X is said to be a *residue of type* Δ^1.

Let Γ be a geometry over I and let J be a subset of I. For any flag X of Γ the set $\tau^{-1}(J) \cap X^*$ is called the J-*shadow* of X and denoted by $Sh_J(X)$. The *incidence system of type* $\Delta_{n,J}$ *(associated with the geometry* Γ *of type* Δ_n*)* is defined to be the incidence system (P,L) where $P = \tau^{-1}(J)$ and $L = \{Sh_J(X) \mid X \text{ is a flag of type } I\setminus J\}$. If $J = \{j\}$, we write $Sh_j(X)$. For $x \in \Gamma$, the expression $Sh_j(x)$ often replaces $Sh_j(\{x\})$. The type $\Delta_{n,j}$ will sometimes be referred to as *punctured* Coxeter *diagram*. A geometry $\Gamma$of type Δ_n is said to be a *building of type* Δ_n if for any two vertices x,y of Γ and any $i \in I$ with $Sh_i(x) \cap Sh_i(y) \neq \emptyset$, there is a flag X contained in $\{x,y\}^*$ such that $Sh_i(x) \cap Sh_i(y) = Sh_i(X)$. We note that this defition is justified by corollary 6 of [13] which states that the buildings as defined above coincide with buildings as defined in [13] and with the weak buildings as (originally) defined in [12]. Buildings in which every flag of corank 1 is contained in at least three maximal flags will be referred to as *thick buildings*

(these are the buildings of [12]).
The present notion of building is presented in a way strongly influenced by Buekenhout (cf.[1],[14]). In fact, a geometry of type Δ_n is a kind of 'diagram geometry', while buildings are geometries satisfying an 'intersection property'.
The isomorphism classes of thick buildings of type $A_n (n \geq 3)$ are parametrized by the isomorphism classes of skew fields, the isomorphism classes of thick buildings of type $D_n (n\geq 4)$, $E_n (n=6,7,8)$ by isomorphism classes of fields (cf. Tits [12]). For a skew field K (a field K) we shall denote by $A_n(K)$ ($D_n(K)$, $E_n(K)$ respectively) the unique thick building (up to isomorphism) of type A_n (D_n,E_n respectively) parametrized by K, i.e. the unique thick building of the given type all of whose residues of type A_2 are isomorphic to the incidence structure $A_2(K)$ of the projective plane defined over K. Thus, for example, $A_n(K)$ for a skew field K, may be viewed as the n-partite graph whose vertex set is the collection of all nonempty proper subspaces of the projective space PG(n,k) of rank n over K and in which two distinct subspaces are incident whenever one of them is contained in the other. The elements of $A_n(K)$ whose type is i correspond to the subspaces of rank i-1 of PG(n,K). Tits [13] has observed that in fact these examples and their 'joins' are essentially the only geometries of type A_n for $n \geq 3$. He also observed that this need not be the case for geometries of arbitrary spherical type.

1.4 We are now in a position to define the spaces of particular interest to our goals. Let Δ_n be a Dynkin diagram (of Figure 1) and let J be a subset of I_n. An incidence system of type $\Delta_{n,J}$ associated with a thick building of type Δ_n is called a *Lie incidence system of type* $\Delta_{n,J}$.
For $\Delta_n = A_n(D_n,E_n)$, K a (skew) field, and J a subset of I_n, we let $\Delta_{n,J}(K)$ stand for the incidence system (unique up to isomorphism) of type $\Delta_{n,J}$ associated with the building $\Delta_n(K)$. Again, for $j \in I_n$, we replace $\Delta_{n,J}$ by $\Delta_{n,j}$, and $\Delta_{n,J}(K)$ by $\Delta_{n,j}(K)$. Thus, $A_{n,j}(K)$ may be identified with the incidence system (P,L) whose points are the subspaces of PG(n,K) of rank j-1 and whose lines are the sets of all members of P containing a subspace X of rank j-2 and contained in a subspace Y of rank j, where $X \subseteq Y$. In particular, $A_{n,1}(K) = PG(n,K)$.
Let us briefly sketch the connection with groups of Lie type. Suppose G is a group of Lie type whose Lie rank n is at least 3. Then G admits a socalled

B,N -pair associated with a Coxeter diagram Δ_n. For definitions and a full account, the reader is (again) referred to the celebrated work [12]. Write $I = I_n$. There is a 1-1 correspondence between the subsets J of I and the subgroups P_J of G containing B such that if $J \subseteq K \subseteq I$, then $P_J \subseteq P_K$. Thus, $P_\emptyset = B$ and $P_I = G$. For J a proper subset of I, let $P = G/P_J$ and $L = \{\{aP_J | aP_J \cap bP_{I\setminus J} \neq \emptyset\} | b \in G\}$. Then (P,L) is a Lie incidence system of type $\Delta_{n,I\setminus J}$ associated with the building Γ of type Δ_n, where Γ is the geometry with $\Gamma_i = G/P_{I\setminus\{i\}}$ for $i \in I$ in which incidence between aP_J and bP_K for $a,b \in G$ and $J,K \subseteq I$ is defined by $aP_J \cap bP_K \neq \emptyset$. The importance of buildings stems from the fact that the converse is true for thick buildings: If Δ_n is a Dynkin diagram of rank at least 3, then for any thick building Γ of type Δ_n there is a group G of Lie type consisting of automorphisms of Γ admitting a B,N -pair such that the geometry derived from G as described in the preceding paragraph coincides with Γ. Tits' work [12] is mainly devoted to proving this result.

The results of Buekenhout-Shult [4], Veldkamp and Tits [12] together, yield that nondegenerate polar spaces of finite rank at least 3 whose lines are thick are Lie incidence systems of type $D_{n,1}$ or $C_{n,1}$. Here, $D_{n,1}$ may be left out, as any building Γ of type $D_{n,1}$ gives rise to a building $C_{n,1}$ in such a way that an incidence system of type $D_{n,1}$ associated with Γ is also a Lie incidence system of type $C_{n,1}$ $(n\geq 3)$. For, given a building Γ of type D_n define Γ' as the union of Γ'_j for $i = 1,2,\ldots,n$, where $\Gamma'_i = \Gamma_i$ for $i = 1,\ldots,n-2$, Γ'_{n-1} is the collection of flags of Γ of type $\{n-1,n\}$, and $\Gamma'_n = \Gamma_{n-1} \cup \Gamma_n$. Let τ' be the type map $\Gamma' \to I_n$ corresponding to the given partition of Γ' and let x,y be incident in Γ' whenever $\tau'(x) \neq \tau'(y)$ and $x \perp_\Gamma y$. Then Γ' is a building of type C_n (which is not thick!) such that any incidence system of type $D_{n,1}$ associated with Γ is an incidence system associated with Γ'.

1.5 Theorems characterizing Lie incidence systems appear in Buekenhout [3], Cameron [5], Cohen [6], Cooperstein [9] and Tallini [11]. The main goal of this paper is to extend this work by the following two theorems. To state the first theorem, however we need the notion of quotient for incidence systems.

DEFINITION. If G is a group of automorphisms of an incidence system (P,L) such that $L \not\subseteq x^G$ for any $x \in P$ and $L \in L$ then the incidence system $(P,L)/G = (P/G, L/G)$ whose points are the orbits of G in P and whose lines are the collections of orbits contained in $\cup_{g\in G} L^g$ for $L \in L$, is called the *quotient of* (P,L) *by* G.

THEOREM 1. *Let* $k \geq 2$ *and let* (P,L) *be a parapolar space of finite singular rank* s. *Then* (P,L) *satisfies* $(P3)_k$ *and* $(P4)$ *if and only if one if one of the following statements holds:*

(i) $k = s$ *and* (P,L) *is a nondegenerate polar space of rank* $k+1$ *with thick lines.*

(iia) $k = 2$, $s \geq 3$, *there is a natural number* $n (4 \leq n \leq 2s-1)$ *and a skew field* K *such that* $(P,L) \cong A_{n,d}(K)$, *where* $d = n-s+1$.

(iib) $k = 2$, $s \geq 5$ *and there is an (infinite) skew field* K *such that* $(P,L) \cong A_{2s-1,s}(K)/\langle\sigma\rangle$, *where* σ *is an involutory automorphism of* $A_{2s-1,s}(K)$ *induced by a polarity of the projective space* $PG(2s-1,K)$ *of Witt-index* $\leq s-5$.

(iii) $k = 3$, $s \geq 4$, *there is a field* K *and there are families* S *and* D, *respectively, of subspaces of* (P,L) *whose members are geodesically closed subspaces of* (P,L) *isomorphic to* $D_{4,1}(K)$ *and* $D_{5,5}(K)$, *respectively, such that any pair* $x,y \in P$ *with* $d(x,y) = 2$ *is contained in a unique member* $S(x,y)$ *of* S *and such that any triple* $x,y,z \in P$ *with* $d(x,y) = 2$, $d(y,z) = 1$ *and* $\{x,y,z\}^\perp$ *a maximal clique in* $\{x,y\}^\perp$, *is contained in a unique member* $D(x,y,z)$ *of* D. *Moreover, for any* $x \in P$, *the incidence system* $(L_x, L_x(V_x))$ *is isomorphic to* $A_{s,2}(K)$.

(iv) $k = 4$, $s = 5$, *and there is a field* K *such that* $(P,L) \cong E_{6,1}(K)$.

(v) $k = 5$, $s = 6$, *and there is a field* K *such that* $(P,L) \cong E_{7,1}(K)$.

Part of the above theorem has also been announced by Professor Shult. Special cases of Theorem 1 provide characterizations of Lie incidence systems of type $D_{5,5}$ and $D_{6,6}$, see Theorem 4 below. The only Lie incidence systems satisfying (iii) of Theorem 1 are those of type $D_{n,n}$. However, letting $K = \mathbb{R}$, there are quotients $D_{n,n}(\mathbb{R})/\langle\sigma\rangle$ of $D_{n,n}(\mathbb{R})$ (for n even, ≥ 10) by involutory automorphisms σ induced by 'polarities' of Witt-index at most $n-10$ of the orthogenal $2n$-dimensional linear space in which $D_n(\mathbb{R})$ can be embedded (commuting with the defining polarity), that also satisfy (iii) but are not of Lie type. Cooperstein [10], has given additional 'global' axioms so as to provide a characterization of Lie incidence systems of type $D_{n,n}$. It would be of interest to know whether any incidence system satisfying the axioms of the above theorem with $k = 3$ is a quotient of a Lie incidence system of type $D_{n,n}$.

THEOREM 2. *Let* $k \geq 3$ *and suppose* (P,L) *is a parapolar space of finite singular rank* s. *Then* (P,L) *satisfies* $(F3)_k$ *and* (F4) *if and only if there exists a field* K *such that one of the following statements holds:*

(i) $k = s$ *and* (P,L) *is a nondegenerate polar space of rank* $k+1$ *with thick lines.*

(ii) $k = 3$, $s = 4$, *and* $(P,L) \cong D_{5,5}(K)$ *or* $E_{6,4}(K)$.

(iii) $k = 4$, $s = 5,6$, *and* $(P,L) \cong E_{6,1}(K)$ *or* $E_{7,7}(K)$.

(iv) $k = 6$, $s = 7$, *and* $(P,L) \cong E_{8,1}(K)$.

The Lie incidence systems $F_{4,1}(K)$, $E_{7,7}(K)$ and $E_{8,1}(K)$ for a field K can all be identified with the natural geometries whose points are the root subgroups (corresponding to roots of a single length) of the underlying Chevalley group. In [6] Lie incidence systems of type $F_{4,1}$ are characterized as parapolar spaces (P,L) in which (F3) and (F4) hold and in which there are no minimal 5-circuits (i.e., if $x_1,x_2,x_3,x_4,x_5 \in P$ with $x_{i+1} \in x_i^{\perp} \setminus \{x_i\}$ for $i = 1,2,3,4,5$, indices taken modulo 5, then $x_i^{\perp} \cap x_{i+2}x_{i+3} \neq \emptyset$). In the Lie incidence systems of type $E_{6,4}$, $E_{7,7}$, $E_{8,1}$, minimal 5-circuits do exist (see [7]).

2. PRELIMINARY RESULTS

2.1. Let us first review the theory of parapolar spaces. See [2], [6], [9] for proofs and details.

PROPOSITION 1. *Let* (P,L) *be a parapolar space. Then*

(i) $L = \{\{x,y\}^{\perp\perp} | x \in P$ *and* $y \in x^{\perp}\backslash x\}$. *In particular,* (P,L) *is a linear incidence system and completely determined by its collinearity graph.*

(ii) *For any* $x,y \in P$ *with* $d(x,y) = 2$ *and* $|\{x,y\}^{\perp}| > 1$, *set* $S(x,y) = \{z \in P | z^{\perp} \cap L \neq \emptyset$ *for any* $L \in L(x^{\perp} \cap y^{\perp})\}$. *Then* $S(x,y)$ *is a* *geodesically closed subspace isomorphic to a nondegenerate polar space. In particular,* $z^{\perp} \cap S(x,y)$ *is a singular (posibly empty) subspace for any* $z \in P \setminus S(x,y)$*. Moreover,* $S(x,y) = \langle \{x,y\} \cup \{x,y\}^{\perp} \rangle$.

(iii) *If* $\{x,y\}^{\perp}$ *is a polar space of rank* k*, then* $S(x,y)$ *has rank* k+1 *(as a polar space).*

(iv) *Each singular subspace of rank at most 2 is contained in* $S(x,y)$ *for suitable* $x,y \in P$. *Hence it is empty, a point, a line or a projective plane.*

(v) *If* M *is a maximal singular subspace, then* M *is a projective space and contains a line properly.*

Note that for any $x,y \in P$ as in (ii) and any $x,y_1 \in S(x,y)$ with $x \notin y_1^{\perp}$, we have $S(x,y) = S(x,y_1)$ as a result of (ii). The family of $S(x,y)$ obtained as in (ii) for a parapolar space (P,L) will be denoted by S. Its members are called *symplecta*. The family of singular subspaces of rank i will be denoted by $V^{(i)}$. Thus $V^{(o)}$ is the collection of singletons of P (often sloppily referred to as 'point'), and $V^{(1)} = L$. Instead of $V^{(2)}$ we shall also write V. Its members are called *planes*. Finally, let M stand for the collection of maximal singular subspaces of (P,L) and put $M^{(i)} = M \cap V^{(i)}$.
The *residue (of a parapolar space* (P,L)*) at point* x of P is the incidence system $P^{x} = (L_x, L_x(V_x))$. If A is an incidence system isomorphic to the residue of (P,L) at x, then (P,L) is said to be *locally* A *at* x. If (P,L) is locally A at every x of P, then we say that (P,L) is *locally* A. Moreover if A is a collection of incidence systems, (P,L) is called *locally* A if for each point x of P there is a member A of A such that (P,L) is locally A at x. Thus 'locally polar' for (P,L) means that (P,L) is locally A where A stands for the collection of spaces.

2.2. The following lemma provides a means to recognize polar spaces locally among parapolar spaces. A first version is to be found in Cooperstein [9].

<u>LEMMA 2.</u> *Let* (P,L) *be a parapolar space such that the residue at any point is connected. Then for each* $x \in P$ *the following statements are equivalent.*

(i) (P,L) *is a polar space.*

(ii) (P,L) *is locally polar.*

(iii) (P,L) *is locally polar at* x.

(iv) *There is exactly one symplecton on* x.

<u>PROOF.</u> Obviously, (i) implies (ii) and (ii) implies (iii). Suppose (iii) holds. By (iv) of Proposition 1, there is a symplecton S on x. Take $y \in S\backslash x^{\perp}$. Then $S = S(x,y)$, and $d(x,y) = 2$. We shall prove that $x^{\perp}$ is contained in S, thus establishing (iv). Thus let $z \in x^{\perp}\backslash\{x\}$. Since $\{x,y\}^{\perp}$ is a nondegenerate polar space of rank at least 2, there is a minimal 4-circuit u_1,u_2,u_3,u_4 (i.e. $u_i \neq u_{i+1}$ and $u_i \in u_{i+1}^{\perp}\backslash u_{i+2}^{\perp}$ for all i, indices modulo 4) of points contained in $\{x,y\}^{\perp}$. Write $V_i = \langle x,u_i,u_{i+1}\rangle$. Then $L(V_i)$ is a line in the residue of x, so there is $N_i \in L_x(V_i)$ with $L \subseteq N_i^{\perp}$ by (iii). Since V_i is a projective plane, there is $v_i \in V_i$ with $\{v_i\} = N_i \cap x_ix_{i+1}$. It results that $z \in \{v_1,v_2,v_3,v_4\}$. But $\{v_1,v_2,v_3,v_4\}$ is not a clique, so by (ii) of Proposition 1, we get $z \in S$. Hence $x^{\perp} \subseteq S$, as wanted.
Finally, we show that (iv) implies (i). Assume (iv) holds and let S be the single symplecton on x. We first claim that $x^{\perp}$ is contained in S. For suppose there is $z \in x^{\perp}\backslash S$. Then by connectedness of the residue at x, there is a path of finite length from z to a point u of $x^{\perp} \cap S\backslash\{x\}$. Reasoning by induction, we may assume that z is actually collinear with u. Since $z^{\perp} \cap S$ is a singular subspace of S, there exists $y \in u^{\perp} \cap S\backslash z^{\perp}$. Now x and u are distinct points of $\{y,z\}^{\perp}$, so that $S(y,z)$ is well defined. Moreover, it is a symplecton containing x and hence $S(y,z) = S$. This yields $z \in S$, proving the claim.

Next, let $y \in x^{\perp}\backslash\{x\}$. Then $y \in S$ as we have just seen. We claim that S is the only symplecton on y. Let L be a line on y. We shall establish by induction on the distance of xy to L within $P^y = (L_y, L_y(V_y))$ that S is the only symplecton on L. In view of the connectedness of the residue at y, this suffices for the proof of the claim.

If $L = xy$, the claim is clearly true. Suppose $L \neq xy$ and let $x_0,x_1,\ldots,x_s$ in $y^{\perp}\backslash\{y\}$ be such that x_0y (=xy), $x_1y,\ldots,x_sy$(=L) is a minimal path in the residue at y from xu to L. Then $s \geq 1$. Assume T is a symplecton on L distinct from S. If $i < s$, then $x_i \in S\backslash T$ by induction. Put $u = x_{s-1}$. Note that $\mathrm{rk}\,(u^{\perp}\cap T) \geq 1$ as $L \subseteq u^{\perp} \cap T$. If $\mathrm{rk}\,(u^{\perp}\cap T) = 1$, take $z_1,z_2 \in (u^{\perp}\cap T)^{\perp} \cap T$ with $z_1 \notin z_2^{\perp}$. Then $u^{\perp} \cap T \subseteq \{u,z_i\}^{\perp}$ for each i, so $S(u,z_i)$ exists and $S = S(u,z_i)$ by induction. It follows that $z_i \in S$ and $T = S(z_1,z_2) = S$.
Suppose $\mathrm{rk}\,(u^{\perp}\cap T) \geq 2$. Let $z \in T\backslash u^{\perp}$. Then $\{z,u\}^{\perp}$ contains $z^{\perp}\cap(u^{\perp}\cap T)$, a sub-

space of $u^\perp \cap T$ of rank at least 1, since T is a polar space. Therefore $z \in S(u,z) = S$, proving $T \backslash u^\perp \subseteq S$. Since clearly $u^\perp \cap T \subseteq S$, we obtain $T \subseteq S$ whence $T = S$. This ends the proof of the claim. The connectedness of (P,L) now yields that for any $y \in P$ the subspace S is the only symplecton on y. Thus $P = S$ and (P,L) is a polar space, whence (i). □

2.3. A *bouquet of (para-) polar spaces* in an incidence system (P,L) containing a point x such that $P\backslash\{x\}$ has more than one connected component and such that the union of any such component with $\{x\}$ forms a subspace which is a (para-) polar space. The requirement that the residue at each point is connected is necessary in the preceding lemma as a bouquet of polar spaces is a parapolar space satisfying (iv) for all but one point, but not (i). Bouquets of parapolar spaces do not satisfy axioms (P3), (F4) given in the introduction. This explains why these bouquets do not appear in Theorems 1 and 2. There is a useful reformulation of axioms (P4) and (F4) in terms of symplecta. Let J be a subset of $\{-1,0,1\}$ and consider the following axiom for a parapolar space (P,L).

$(F4)_J$ For any symplecton S and point x of $P\backslash S$, the rank of the singular subspace $x^\perp \cap S$ is either a member of J or the singular rank of S.

Now $(F4)_{\{-1,1\}}$ is equivalent to (F4), whereas $(F4)_{\{-1,0\}}$ is equivalent to (P4), so that we have indeed obtained reformulations of (P4) and (F4). The usefulness of these axioms is their behaviour under taking residues. This is explained in the lemma below. First, however, we introduce some more notation. If (P,L) is a Gamma space, $x \in P$ and X a subspace of P containing x, denote by X^x the subspace $L_x(X)$ of P^x (cf.[9]). Note that this is consistent with the notation for P^x for $X = P$. If F is a family of subspaces of P, denote by F^x the family $L_x(F_x) = \{F^x | F \in F_x\}$ of subspaces of P^x.

Lemma 3. *Let* (P,L) *be a parapolar space of singular rank* s. *As usual, let* M,S *respectively stand for the collection of maximal singular subspaces and the collection of symplecta in* (P,L). *Then the following holds for any* $x \in P$.

(i) M^x *is the collection of maximal singular subspaces of* P^x. *If* $M \in M$ *is isomorphic to* $A_{n,1}(K)$ *for some* $n \in \mathbb{N}$, *and some field* K, *then* M^x *is isomorphic to* $A_{n-1,1}(K)$. *In particular,* P^x *has singular rank* s-1.

(ii) *If* $(P,\mathcal{L})$ *satisfies* $(F3)_I$ *for* $I = \{k\in\mathbb{N} \mid k\geq 3\}$, *and* P^x *is connected, then* P^x *is a parapolar space satisfying* (P3) *whose collection of symplecta is* S^x *(which is in bijective correspondence with* $\mathcal{S}_x$*).*

(iii) *If* $(P,\mathcal{L})$ *satisfies* (P4) *and* $(P3)_k$ *for some* $k \geq 3$, *and* $x \in P$, *then* P^x *satisfies* $(F4)_{\{-1\}}$ *and* $(P3)_{k-1}$ *and has diameter* 2.

(iv) *If* $(P,\mathcal{L})$ *satisfies* (F4) *and* $(F3)_k$ *for some* $k \geq 3$, *and* $x \in P$, *then* P^x *satisfies* $(F4)_{\{0\}}$ *and has diameter* 3 *or* 2 *depending on whether there exist* $S \in \mathcal{S}_x$ *and* $y \in x^\perp\backslash S$ *with* $\mathrm{rk}(y^\perp\cap S) = 1$ *or not.*

PROOF. Since the proof of most statements is straightforward, we shall only treat (ii). So let I be as in (ii) and assume P^x is connected. As (F1) clearly holds, we proceed to prove (F2). Let $L_1,L_2 \in \mathcal{L}_x$ with $L_1 \subseteq L_2^\perp$. We need to show the existence of $L_3,L_4 \in \mathcal{L}_x$ with $L_3 \not\subseteq L_4^\perp$ and $L_i \subseteq L_j^\perp$ for all $i = 1,2$ and $j = 3,4$. Since $\langle L_1\cup L_2\rangle$ is a singular subspace of $(P,\mathcal{L})$ of rank at most 2, Proposition 1 (iv) yields the existence of a symplecton, S say containing both L_1 and L_2, hence x. By familiar properties of nondegenerate polar spaces, there exist $L_3,L_4 \in \mathcal{L}_x(S)$ with $L_3 \not\subseteq L_4^\perp$ and $L_i \subseteq L_j^\perp$ for all $i = 1,2$ and $j = 3,4$. This establishes (F2) for P^x. Next suppose $L_1,L_2 \in \mathcal{L}_x$ have distance 2 in P^x. Then there is $L \in \mathcal{L}_x$ with $L \subseteq L_1^\perp \cap L_2^\perp$. Take $z_i \in L_i\backslash\{x\}$ for $i = 1,2$. Now $\{z_1,z_2\}^\perp$ contains L, so z_1,z_2 is a symplectic pair of P.
Let $\perp_x$ denote collinearity in P^x. Now $L_1^{\perp_x} \cap L_2^{\perp_x} = \mathcal{L}_x(L_1^\perp\cap L_2^\perp) = \mathcal{L}_x(z_1^\perp\cap z_2^\perp)$ is the residue at x of the the nondegenerate polar space $z_1^\perp \cap z_2^\perp$ of rank ≥ 3. Therefore, $L_1^{\perp_x} \cap L_2^{\perp_x}$ is a nondegenerate polar space of rank ≥ 2. This proves (P3) for P^x. Since P^x is connected by assumption and lines of P^x have the same cardinality as the members of $\mathcal{L}$, we conclude that P^x is a parapolar space satisfying (P3).
Finally, we assert that the subspace $S(L_1,L_2)$ of P^x (spanned by L_1,L_2 and $L_1^{\perp_x}\cap L_2^{\perp_x}$) is the residue of the symplecton $S(z_1,z_2)$. Clearly, since $S(z_1,z_2) = \langle\{z_1,z_2\}\cup\{z_1,z_2\}^\perp\rangle$, the residue of $S(z_1,z_2)$ contains $S(L_1,L_2)$. On the other hand, let $L_3\in \mathcal{L}_x(S(z_1,z_2))$. Then there are $z_3,z_4 \in S(z_1,z_2) \cap x^\perp$ with $z_3 \notin z_4^\perp$, and $z_i \in z_j^\perp$ for all $i = 1,2$ and $j = 3,4$ such that $L_3 \subseteq z_3^\perp \cap z_4^\perp$ due to the structure of the nondegenerate polar space $S(z_1,z_2)$ of rank ≥ 4. Thus $L_3\in (xz_3)^{\perp_x} \cap (xz_4)^{\perp_x}$ whilst $xz_3,xz_4 \in L_1^{\perp_x} \cap L_2^{\perp_x} \subseteq S(L_1,L_2)$ and $xz_3^\perp \notin (xz_4)^{\perp_x}$.

It follows from the geodesic closure of $S(L_1,L_2)$ that L_3 is a member of $S(L_1,L_2)$. This proves the assertion.
The conclusion is that the map $\mathcal{S}_x \to \mathcal{S}^x$ given by $S \mapsto S^x$ is a bijection from the set of symplecta on x onto the set of symplecta in P^x. This proves (ii).
□

2.4. The following lemma is of similar use as Lemma 2 for local recognition of the incidence system.

LEMMA 4. *Let* (P,L) *be a parapolar space satisfying* (P3). *Suppose* X *is a geodesically closed subspace of* P. *If* $x \in P$ *with* $x^\perp \subseteq X$, *then* $X = P$.

PROOF. Let $y \in P$. We show that $y \in X$ by induction on $d(x,y)$.
If $d(x,y) \le 1$, then clearly $y \in X$. Suppose $d(x,y) > 1$. Then there is a point z of $x^\perp$ such that $d(x,z) = d(x,y)-1$. Thanks to induction, it suffices to show $z^\perp \subseteq X$. Suppose $u \in z^\perp$. Then $d(x,u) \le 2$. If $u \in x^\perp$, then $u \in X$ as we have seen before. If $d(x,u) = 2$, then due to (P3) we can find $v,w \in x^\perp \cap u^\perp$ with $v \notin w^\perp$. Now $u \in S(u,x) = S(v,w) \subseteq X$ since X is geodesically closed. This yields $z^\perp \subseteq X$ as wanted. □

2.5. The following lemma shows that under suitable assumptions on the parapolar space is question, there is a unique (skew) field K such that all singular subspaces are projective spaces over K.

LEMMA 5. *Let* $k \ge 3$. *Suppose* (P,L) *is a parapolar space of finite singular rank* s *satisfying either* $(P3)_k$ *and* (P4), *or* $(F3)_k$ *and* (F4). *If* (P,L) *is not a polar space, then the following statements hold.*

(i) *The relation* $\approx$ *on* M *defined by* $M_1 \approx M_2$ *for* $M_1,M_2 \in M$ *if and only if* $rk(M_1 \cap M_2) = k-2$ *turns* $(M,\approx)$ *into a graph having at most two connected components. Moreover* $M_1 \approx M_2$ *implies that* M_1 *and* M_2 *are isomorphic subspaces.*

(ii) *There exists a field* K *and a number* $t \ge k$ *such that any* $M \in M$ *is isomorphic to* $PG(m,k)$ *for some* $m \in \{s,t\}$ *and such that any symplecton is isomorphic to* $D_{k+1,1}(K)$.

(iii) *If* $k = 3$ *then there is a field* K *such that* (P,L) *is either locally*

$A_{s,2}(K)$ *or locally* $A_{5,3}(K)$ *according as* (P4) *holds or not.*

PROOF. We use induction on k.

(i) Let N be a singular subspace of P of rank k-1. We first show that any $M \in \mathcal{M}$ is connected within $(\mathcal{M},\approx)$ to a member of $\mathcal{M}_N$. Thanks to connectivity of (P,L) it suffices to show this holds for $M \in \mathcal{M}$ with $M \cap N \neq \emptyset$. Suppose, therefore, $x \in M \cap N$ and consider M^x. Due to Lemma 3, M^x is a maximal singular subspace of the parapolar space P^x for which $(P3)_{k-1}$ and $(F4)_{\{0\}}$ holds.
If $k=2$, then P^x is a Lie incidence system of type $A_{5,3}$ or $A_{n,2}$ or a quotient of such an incidence system by [8], so M^x is connected within P^x to a maximal singular subspace of P^x containing N^x.
If $k=3$, it follows, again from [8], that the graph on $\mathcal{M}^x$ in which M_1^x, M_2^x are connected if and only if $rk(M_1^x \cap M_2^x) = 0$ has at most two connected components and that adjacent members are isomorphic. In view of the first statement, this yields that $(\mathcal{M},\approx)$ has at most two connected components. Also, it is immediate from thickness of lines that $M_1 \approx M_2$ for $M_1, M_2 \in \mathcal{M}$ implies that M_1 is isomorphic to M_2.
For $k>3$, the same statements for the residue follow from induction, while the transition from the residue to $(\mathcal{M},\approx)$ is identical. This ends the proof of (i).

(ii) Let $x \in P$. If $k=3$, then there is a skew field K and a number $t \geq 3$ such that any $M \in \mathcal{M}_x$ satisfies $M^x \cong PG(m,K)$ for some $m \in \{s-1,t-1\}$, and such that any $S \in \mathcal{S}_s$ satisfies $S^x \cong A_{3,2}(K)$. According to Lemma 3, this yields $M \cong PG(m+1,K)$ and $S \cong D_{4,1}(K)$, as $D_{4,1}(K)$ is the only non-degenerate polar space with thick lines containing a point at which the residue is isomorphic to $A_{3,2}(K)$. Moreover, this implies (cf. Tits [12], 6.12) that K is a field.
Now let $M_1 \in \mathcal{M}$ and $S_1 \in \mathcal{S}$, and take $y \in S_1$. According to (i) we have $M_1 \cong PG(m,k)$ for some $m \in \{s,t\}$. Thus members of $\mathcal{M}$ are defined over K. Furthermore, reasoning for y as for x above, we obtain a field K_1, such that $S \cong D_{k+1,1}(K_1)$. But the maximal singular subspaces of S are subspaces of members of $\mathcal{M}$, hence defined over K. The conclusion is that K_1 coincides with K.
For $k>3$, the induction hypothesis yields a field K and a number

$t \geq k$ such that any $M \in M_x$ and $S \in S_x$ satisfy $M^x \cong PG(m-1,K)$ and $S^x \cong D_{k,1}(K)$ for some $m \in \{s-1,t-1\}$. Now statement (ii) follows as for $k = 3$.

(iii) Let $k = 3$, and pick $x \in P$. According to Lemma 3, P^x is a parapolar space satisfying $(P3)_2$ and either $(F4)_{\{-1\}}$ of $(F4)_{\{0\}}$. But the latter two axioms are easily seen to be equivalent to (Q4) and (R4) of [8] respectively. Therefore by the applications of [8] and Lemma 3, there is a skew field K such that either $s \geq 4$ and $P^x \cong A_{s,2}(K)$ or $s = 3$ and $P^x \cong A_{5,3}(K)$. Due to (ii) and the fact that for each value of s the residue on x is uniquely determined up to isomorphism, we obtain that K is a field and that (P,L) is locally $A_{s,2}(K)$ or locally $A_{5,3}(K)$. This finishes the proof of the lemma. □

2.6. The part of Theorem 1 concerning $A_{n,d}(K)$ has been dealt with in [8], [10]. The idea of the proof of Theorem 1 is to establish by induction on k what (P,L) is locally isomorphic to, and use these local data to recover all subspaces of P that will occur as verties of the geometry (of type the prevailing Coxeter diagram) to be associated to (P,L). In order to conclude that this geometry is a building we shall make use of the following result of Tits ([13], Proposition 9).

Let Γ be a geometry of type $\Delta_n = D_n$ $(n\geq 4)$, E_n $(n=6,7,8)$ and consider the following assertions for $2 \leq i \leq n-1$.

(LL) If $\gamma_2,\gamma_2' \in \Gamma_2$ are both incident to $\gamma_1,\gamma_1' \in \Gamma_1$ and $\gamma_1 \neq \gamma_1'$, then $\gamma_2 = \gamma_2'$.

(LH) If $\gamma_2 \in \Gamma_2$ and $\gamma_n \in \Gamma_n$ are both incident to two distinct vertices of Γ_1, then $\gamma_2 * \gamma_n$.

(HH) If two distinct vertices of Γ_n are both incident to two distinct verties $\gamma_1,\gamma_1' \in \Gamma_1$, then there is $\gamma_2 \in \Gamma_2$ such that $\gamma_1 * \gamma_2$ and $\gamma_1' * \gamma_2$.

$(0)_i$ If $\gamma_i,\gamma_i' \in \Gamma_i$ have the same shadow on Γ_1 (i.e., $Sh_1(\gamma_i')=Sh_1(\gamma_i)$), then $\gamma_i = \gamma_i'$.

<u>THEOREM 3</u> (Tits). *Let Γ be a geometry of type Δ_n.*

(i) *If $\Delta_n = D_n$, then Γ is a building if and only if it satisfies $(0)_i$ for*

$i = 2,3,\ldots, n-2$ *and* (LL).

(ii) *If* $\Delta_n = E_6$, *then* Γ *is a building if and only if it satisfies* $(0)_i$ *for* $i = 2,3$ *and* (LL).

(iii) *If* $\Delta_n = E_7$, *then* Γ *is a building if and only if it satisfies* $(0)_i$ *for* $i = 2,3,4$, (LL) *and* (LH).

(iv) *If* $\Delta_n = E_8$, *then* Γ *is a building if and only if it satisfies* $(0)_i$ *for* $i = 2,3,4,5$, (LL), (LH) *and* (HH).

3. PROPERTIES OF SOME LIE INCIDENCE SYSTEMS

As most of the properties given are easily checked, we shall virtually give no proofs.

3.1. The next proposition deals with the 'only if' part of Theorem 1 and 2.

PROPOSITION 2. (i) *Let* $n \geq 4$ *and let* $2 \leq i \leq \frac{n+1}{2}$.

(i) *If* K *is a skew field, then* $A_{n,i}(K)$ *is a parapolar space of diameter* i *and of singular rank* $n-i+1$, *satisfying* $(P3)_2$ *and* (F4). *Axiom* $(F4)_{\{-1\}}$ *holds for* $A_{n,i}(K)$ *if and only if* $i = 2$; *and axiom* $(F4)_{\{0\}}$ *holds if and only if* $(n,i) = (5,3),(4,2)$.

(ii) *Assume* $n \geq 5$ *and let* K *be a field. Then* $D_{n,n}(K)$ *is a parapolar space of diameter* $[\frac{n}{2}]$ *and of singular rank* $n-1$ *satisfying* $(P3)_3$ *and* (F4). *Axiom* $(F4)_{\emptyset}$ *holds if and only if* $n = 5$, *and axiom* $(F4)_{\{-1\}}$ *holds if and only if* $n = 5$ *or* 6.

(iii) *Let* K *be a field.* $E_{6,1}(K)$, $E_{6,4}(K)$, $E_{7,1}(K)$, $E_{7,7}(K)$, $E_{8,1}(K)$ *are parapolar spaces having the properties indicated in the table below.*

PROOF. (i) and (ii) are easily established by use of the "classical model" for the associated buildings (compare [10]). (iii) can be proved by means of a reduction (using the B,N-pair) to the analogous statement for the corresponding Weyl group in which case the verification is straightforward.

□

TABLE 2

$\Delta_{n,i}$	axioms	diameter	singular rank	isomorphism type of Symplecta
$E_{6,1}(K)$	$(P3)_4$, $(F4)_{\{-1\}}$	2	5	$D_{5,1}(K)$
$E_{6,4}(K)$	$(F3)_3$, $(F4)$	3	4	$D_{4,1}(K)$
$E_{7,1}(K)$	$(P3)_5$, $(F4)_{\{0\}}$	3	6	$D_{6,1}(K)$
$E_{7,7}(K)$	$(F3)_4$, $(F4)$	3	6	$D_{5,1}(K)$
$E_{8,1}(K)$	$(F3)_6$, $(F4)$	3	7	$D_{7,1}(K)$

3.2. PROPERTIES OF SOME GRASSMANNIANS

PROPOSITION 3. *Let* $(P,L) = A_{s,2}(K)$ *for some* $s \geq 4$ *and some skew field* K. *Then*

(i) *Diameter* $(P,L) = 2$.

(ii) (P,L) *satisfies* $(P3)_2$ *and* $(F4)_{\{-1\}}$.

(iii) $(F4)_\emptyset$ *holds for* (P,L) *if and only if* $s = 4$, *and* $(F4)_{\{-1\}}$ *holds if and only if* $s \leq 5$.

(iv) *If* $S,T \in S$ *and* $S \neq T$, *then* $\mathrm{rk}(S\cap T) = -1,0,2$. *Moreover, if* $\mathrm{rk}(S\cap T) = 2$, *then* $\langle S\cup T\rangle$ *is a geodesically closed subspace isomorphic to* $A_{4,2}(K)$.

(v) *We have* $\mathrm{rk}(S\cap T) = 0,2$ *for all* $S,T \in S$ *with* $S \neq T$ *if and only if* $n \leq 5$.

(vi) *If* $L \in L$, $V \in V$ *and* $S_1,S_2,S_3 \in S_V$ *are such that* $S_i \cap L$ $(i=1,2,3)$ *are three distinct points of* L, *then* $\langle S_1\cup S_2\cup S_3\rangle = \langle S_1\cup S_2\rangle \cong A_{4,2}(K)$.

(vii) *If* $S_1,S_2,S_3 \in S$ *satisfy* $S_i \cap S_j \in V$ *and* $S_i \cap S_j \neq S_i \cap S_k$ *whenever* $\{i,j,k\} = \{1,2,3\}$, *then* $\langle S_1\cup S_2\cup S_3\rangle = \langle S_1\cup S_2\rangle \cong A_{4,2}(K)$.

(viii) *If* D *is a subspace of* (P,L) *isomorphic to* $A_{4,2}(K)$ *there is no* $z \in P$ *such that* $z^\perp \cap S \in V^{(2)}$ *for all* $S \in S(D)$.

PROPOSITION 4. *Let* $(P,L) = A_{4,2}(K)$ *for some skew field* K. *Then*

(i) *If* $S,T \in \mathcal{S}$ *with* $S \neq T$, *then* $rk(S \cap T) = 2$.

(ii) *Let* $V \in \mathcal{V}$ *and* $S_1, S_2 \in \mathcal{S}_V$. *For any two distinct* $x,z \in V$ *there are distinct collinear* $u_1 \in x^\perp \cap S_1$ *and* $u_2 \in x^\perp \cap S_2$ *such that* $z \in u_1u_2$.

(iii) *For any* $S \in \mathcal{S}$ *and* $x \in P\backslash S$, *we have* $P = \langle x,S \rangle$.

<u>PROPOSITION 5.</u> *Let* $(P,\mathcal{L}) = A_{5,3}(K)$ *for some skew field* K. *Then*

(i) *diameter* $(P,\mathcal{L}) = 3$.

(ii) $(P,\mathcal{L})$ *satisfies* $(P3)_2$ *and* $(F4)_{\{0\}}$.

(iii) *If* $S,T \in \mathcal{S}$ *with* $S \neq T$, *then* $rk(S \cap T) = -1,1,2$. *If* $rk(S \cap T) = 2$, *then* $\langle S \cup T \rangle$ *is a geodesically closed subspace isomorphic to* $A_{4,2}(K)$. *Moreover, any subspace isomorphic to* $A_{4,2}(K)$ *can be obtained in this way.*

(iv) *If* $V \in \mathcal{V}$ *and* S_1,S_2,S_3 *are three distinct symplecta containing* V, *then* $\langle S_1 \cup S_2 \cup S_3 \rangle = \langle S_1 \cup S_2 \rangle \cong A_{4,2}(K)$.

(v) *If* $V \in \mathcal{V}$, $a_1,a_2 \in P$ *and* $S_1,S_2 \in \mathcal{S}_V$ *satisfy* $a_1 \notin a_2^\perp$, $a_1 \in S_1\backslash S_2$ *and* $a_2 \in S_2\backslash S_1$ *then for any* $c \in a_1^\perp \cap a_2^\perp\backslash(S_1 \cup S_2)$ *we have* $rk(c^\perp \cap S_i) = 3$ *for both* $i = 1,2$.

(vi) *If* $S_1,S_2,S_3 \in \mathcal{S}$ *satisfy* $S_i \cap S_j \in \mathcal{V}$ *and* $S_i \cap S_j \neq S_i \cap S_k$ *whenever* $\{i,j,k\} = \{1,2,3\}$, *then* $\langle S_1 \cup S_2 \cup S_3 \rangle = \langle S_1 \cup S_2 \rangle \cong A_{4,2}(K)$.

(vii) *If* D_1,D_2 *are distinct subspaces isomorphic to* $A_{4,2}(K)$, *then either* $D_1 \cap D_2 \in \mathcal{S}$ *or* $D_1 \cap D_2$ *is a singular subspace of rank* -1 *or* 3.

(viii) *If* $S_1,S_2 \in \mathcal{S}$, $S_1 \neq S_2$ *and* $x \in S_1 \cap S_2$, $z_i \in S_i\backslash x^\perp$ *for each* $i = 1,2$ *with* $d(z_1,z_2) = 1$, *then* $rk(S_1 \cap S_2) = 2$.

(ix) *If* D_1,D_2,D_3 *are distinct subspaces isomorphic to* $A_{4,2}(K)$ *such that* $D_1 \cap D_2$, $D_2 \cap D_3 \in \mathcal{M}^{(3)}$ *then* $D_1 \cap D_3 \in \mathcal{M}^{(3)}$ *and* $rk(D_1 \cap D_2 \cap D_3) = 0,3$.

(x) *If* D,D_1,D_2,D_3 *are distinct subspaces isomorphic to* $A_{4,2}(K)$ *such that* $D_1 \cap D_2 \cap D_3 \in \mathcal{M}^{(3)}$ *and* $D \cap D_i \in \mathcal{M}^{(3)}$ *for* $i = 1,2$, *then* $D \cap D_3 \in \mathcal{M}^{(3)}$.

(xi) *If* D_1,D_2 *are distinct subspaces isomorphic to* $A_{4,2}(K)$ *and* $M_i \in \mathcal{M}(D_i)$ $(i=1,2)$ *with* $rk(D_1 \cap D_2) = 3$ *and* $rk(M_1 \cap M_2) \geq 0$, *then there is a subspace* D *isomorphic to* $A_{4,2}(K)$ *such that* D *contains* $M_1 \cup M_2$.

3.3. PROPERTIES OF $D_{n,n}(K)$ FOR n = 5,6.

PROPOSITION 6. *Let* $(P,L) = D_{5,5}(K)$ *for some field* K. *Then*

(i) *diameter* $(P,L) = 2$.

(ii) (P,L) *satisfies* $(P3)_3$ *and* $(F4)_\emptyset$.

(iii) $S,T \in S$, $S \neq T \Rightarrow rk(S \cap T) = -1,3$ *and* $\langle S,T \rangle = P$.

(iv) $S \in S \Rightarrow S \cong D_{4,1}(K)$.

(v) $M = M^{(3)} \cup M^{(4)}$.

(vi) $x \in P$, $M \in M^{(4)}$, $x \notin M \Rightarrow rk(x^\perp \cap M) = -1,2$.

(vii) $x \in P$, $M \in M^{(3)}$, $x \notin M \Rightarrow rk(x^\perp \cap M) = 0,2$.

(viii) $M_1,M_2 \in M^{(3)} \Rightarrow rk(M_1 \cap M_2) = -1,0,1,3$.

(ix) $M_1,M_2 \in M^{(4)} \Rightarrow rk(M_1 \cap M_2) = -1,1,4$.

(x) $M_1 \in M^{(3)}$, $M_2 \in M^{(4)} \Rightarrow rk(M_1 \cap M_2) = -1,0,2$.

(xi) *If* $M_1,M_2 \in M^{(3)}$ *and* $M_1 \cap M_2 = \emptyset$, *then* $rk(x^\perp \cap M_2) = 2$ *for all* $x \in M_1$.

(xii) *If* M *is a singular subspace such that* $M \cap T \neq \emptyset$ *for any symplecton* T *of* S, *then* $rk(M) = 4$.

(xiii) *If* $M_1,M_2 \in M^{(4)}$ *and* $M_1 \cap M_2 = \emptyset$, *then* $rk(x^\perp \cap M_2) = -1,2$ *for all* $x \in M_1$. *Moreover* $\{z \in M_1 \mid z^\perp \cap M_2 \neq \emptyset\}$ *is a singular subspace of rank* 3.

PROPOSITION 7. *Let* $(P,L) = D_{6,6}(K)$ *for some field* K. *Then*

(i) *diameter* $(P,L) = 3$.

(ii) (P,L) *satisfies* $(P3)_3$ *and* $(F4)_{\{0\}}$, *but* $(F4)_\emptyset$ *does not hold.*

(iii) $S,T \in S$, $S \neq T \Rightarrow rk(S \cap T) = -1,1,3$.

(iv) $S \in S \Rightarrow S \cong D_{4,1}(K)$.

(v) $M = M^{(3)} \cup M^{(5)}$.

(vi) $M \in M^{(5)}$, $x \in P \setminus M \Rightarrow rk(x^\perp \cap M) = -1,2$.

(vii) $M \in M^{(3)}$, $x \in P \setminus M \Rightarrow rk(x^\perp \cap M) = -1,0,2$.

(viii) $M_1,M_2 \in M^{(3)} \Rightarrow rk(M_1 \cap M_2) = -1,0,1,3$.

(ix) $M_1,M_2 \in M^{(5)} \Rightarrow rk(M_1 \cap M_2) = -1,1,5$.

(x) $M_1 \in \mathcal{M}^{(5)}$, $M_2 \in \mathcal{M}^{(3)} \Rightarrow \mathrm{rk}(M_1 \cap M_2) = -1,0,2$.

(xi) *If* $S_1, S_2 \in \mathcal{S}$, $S_1 \neq S_2$ *and* $x \in S_1 \cap S_2$, $z_i \in S_i \backslash x^\perp$ $(i=1,2)$ *such that* $d(z_1,z_2) = 1$, *then* $\mathrm{rk}(S_1 \cap S_2) = 3$

(xii) *If* $S_1, S_2 \in \mathcal{S}$, $S_1 \neq S_2$ *with* $\mathrm{rk}(S_1 \cap S_2) = 3$, *then* $\langle S_1, S_2 \rangle \cong D_{5,5}(K)$.

4. PROOF OF THE MAIN RESULTS

4.1. A CHARACTERIZATION OF $D_{5,5}(K)$.

LEMMA 6. *Let* $(P,\mathcal{L})$ *be a parapolar space of finite singular rank satisfying* $(P3)_3$ *and* $(F4)_{\{-1\}}$. *Then one of the following holds*

(i) $(P,\mathcal{L})$ *is a polar space of rank* 4.

(ii) $(P,\mathcal{L}) \cong D_{5,5}(K)$ *for some field* K.

PROOF. Let $x \in P$. Then P^x satisfies $(P3)_2$ and $(F4)_\emptyset$ and has finite singular rank (cf. Lemma 3). According to [8], this implies that P^x is either a polar space or of type $A_{4,2}$. If P^x is a polar space then $(P,\mathcal{L})$ is a polar space of rank 4 in view of Proposition 1. Therefore, we may assume that $(P,\mathcal{L})$ is locally of type $A_{4,2}$. In particular, by Lemma 5, there is a field K such that maximal singular subspaces of $(P,\mathcal{L})$ are isomorphic to $A_{m,1}(K) = PG(m,K)$ for $m = 3,4$ and symplecta are isomorphic to $D_{4,1}(K)$.

Consider the 5-partite looped graph $(\Gamma,*)$ with type map $\tau : \Gamma \to I_5$ given by $\Gamma_i = \tau^{-1}(i)$, where $\Gamma_1 = \mathcal{S}$, $\Gamma_2 = \mathcal{M}^{(3)}$, $\Gamma_3 = \mathcal{L}$, $\Gamma_4 = \mathcal{M}^{(4)}$, $\Gamma_5 = P$, and in which incidence $\gamma_i * \gamma_j$ for $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$ is given by $\gamma_i \subseteq \gamma_j$ or $\gamma_j \subseteq \gamma_i$ for $\{i,j\} \neq \{1,4\}$, $\{2,4\}$ and by $\mathrm{rk}(\gamma_i \cap \gamma_j) = |i-j|$ otherwise. Then it is readily verified that Γ is a geometry of type D_5. Furthermore, Γ satisfies (LL) and $(O)_i$ for $i = 2,3$.

For, (LL) states that two distinct members of $\mathcal{M}^{(3)}$ determine at most one symplecton, a well known fact due to Proposition 1. Furthermore $(O)_2$ (resp $(O)_3$) states that for any two distinct $M_1, M_2 \in \mathcal{M}^{(3)}$ (respectively for any two distinct $L_1, L_2 \in \mathcal{L}$) there exists $S \in \mathcal{S}$ with $M_i \subseteq S$ and $M_j \not\subseteq S$ (respectively, with $L_i \subseteq S$ and $L_j \not\subseteq S$), where $\{i,j\} = \{1,2\}$. If $M_1 \cap M_2 \neq \emptyset$ (respectively $L_1^\perp \cap L_2^\perp \neq \emptyset$), this clearly follows from consideration of the residue at a point of $M_1 \cap M_2$ (respectively $L_1^\perp \cap L_2^\perp$). If $M_1 \cap M_2 = \emptyset$, then there are $x_i \in M_i$ for $i = 1,2$ with $d(x_1,x_2) = 2$. Thus if $T \in \mathcal{S}$ contains $M_1 \cup M_2$ then $T = S(x_1,x_2)$. Since there is more than one symplecton containing M_1 (by consideration of P^{x_1}), there is $S \in \mathcal{S}$ with $M_1 \subseteq S$ and $M_2 \nsubseteq S$, proving $(0)_2$.

As to $(0)_3$, if $L_1^\perp \cap L_2^\perp = \emptyset$ then there is at most one symplecton containing $L_1 \cup L_2$, whereas there are at least two symplecta containing L_1, so that $(0)_3$ results.

Application of Theorem 3 yields that Γ is a weak building of type D_5. But since $(P,\mathcal{L})$ is locally $A_{4,2}(K)$, the building is thick and defined over K, i.e. $\Gamma \cong D_5(K)$ (cf. [12], p.131). It follows that $(P,\mathcal{L}) \cong D_{5,5}(K)$. This proves Lemma 6. □

4.2. A CHARACTERIZATION OF $D_{5,5}(K)$ AND $D_{6,6}(K)$

Part of the following proposition can be found in [10]. For ease of reference however the proof given below is self-contained.

PROPOSITION 8. *Let* $s \in \mathbb{N}$, $s \geq 4$, *let* K *be a field and suppose that* $(P,\mathcal{L})$ *is a parapolar space but not a polar space, which is either locally* $A_{s,2}(K)$ *or locally* $A_{5,3}(K)$. *Then there is a collection* $\mathcal{D}$ *of geodesically closed subspaces of* $(P,\mathcal{L})$ *isomorphic to* $D_{5,5}(K)$ *such that for any pair* x,X *consisting of a point* $x \in P$ *and a subspace* X *of* P *with* $x \in X \subseteq x^\perp$ *and* $X^x \cong A_{4,2}(K)$, *there is unique member* $D(X)$ *of* $\mathcal{D}$ *containing* X.

PROOF. Note that in view of Lemma 3 $(P,\mathcal{L})$ satisfies $(P3)_3$ and $(P4)$ if it is locally $A_{s,2}(K)$ and that $(P,\mathcal{L})$ satisfies $(F3)_3$ and $(F4)$ if it is locally $A_{5,3}(K)$. Therefore, Lemma 5 applies. Thus any symplecton is isomorphic to $D_{4,1}(K)$.

For a point x in P and a subspace X with $x \in X \subseteq x^\perp$ such that X^x is a subspace of P^x isomorphic to $A_{4,2}(K)$, we introduce the following subsets of $\mathcal{S}$ and P respectively

$$\mathcal{S}[X] = \{S(y,z) \mid y \in X \setminus \{x\},\ z \in X \setminus y^\perp\}.$$

$$D(X) = \bigcup_{S \in \mathcal{S}[X]} S.$$

Note that $\mathcal{S}[X]$ is well defined as indeed any pair $y,z \in X$ with $d(y,z) = 2$ is a symplectic pair. We shall establish that $D(X)$ is a geodesically closed subspace of $(P,\mathcal{L})$ with $D(X)^z \cong A_{4,2}(K)$ for any $z \in D(X)$.

First of all, observe that

$$x^{\perp} \cap D(X) = X,$$

as X^x is geodesically closed, and that for $y \in X \setminus \{x\}$:

$$y^{\perp} \cap D(X) = \bigcup_{z\in X\setminus y^{\perp}} y^{\perp} \cap S(y,z) = \bigcup_{z\in X\setminus y^{\perp}} y^{\perp} \cap z^{\perp},$$

as $(F4)_{\emptyset}$ holds for X^x (cf. Proposition 3).

We proceed in three steps.

(1) $D(X)$ <u>is a subspace of</u> $(P,\mathcal{L})$.

Let a_1,a_2 be distinct collinear points of $D(X)$, and take $b \in a_1a_2\setminus\{a_1,a_2\}$. If both a_1 and a_2 belong to a symplecton contained in $\mathcal{S}[X]$, there is nothing to prove. Thus we may, and shall, restrict to the case where $a_1a_2 \cap x^{\perp} = \emptyset$. Choose $S_i \in \mathcal{S}[X]$ such that $a_i \in S_i$ $(i=1,2)$, and set $M = S_1 \cap S_2$. Then M is a singular subspace of rank 3 on x as any two distinct symplecta of X^x (isomorphic to $A_{4,2}(K)$) meet in a singular subspace of rank 2(cf. Proposition 4(i)).

On the other hand $a_i^{\perp} \cap S_{i+1}$ contains the singular space $\langle a_{i+1}, a_i^{\perp} \cap M\rangle$ of rank 3 (indices taken modulo 2), so that $a_1^{\perp} \cap M = a_2^{\perp} \cap M$. Thus $b^{\perp} \cap M = a_1^{\perp} \cap M$ is a singular subspace of $b^{\perp} \cap x^{\perp}$ and $M \subseteq S(b,x)$. Set $S = S(b,x)$, take $y \in b^{\perp} \cap M$ and consider P^y. Each of the three symplecta S_1^y, S_2^y, S^y of P^y contains the plane M^y and meets the line $\langle a_1,a_2,y\rangle^y$ of P^y disjoint from M^y in a point. Since $P^y \cong A_{s,2}(K)$ or $A_{5,3}(K)$ this yields by Proposition 3(vi) and 5(iv) that $\langle S_1^y,S_2^y,S^y\rangle = \langle S_1^y,S_2^y\rangle \cong A_{4,2}(K)$.

Next, take $z \in (xy)^{\perp} \cap S$. Since $yz \in \langle S_1^y,S_2^y\rangle \cong A_{4,2}(K)$, we get from Proposition 4(ii) that there are distinct coplanar lines $L_i \in \mathcal{L}(S_i)_y$ contained in $x^{\perp}$ $(i=1,2)$ such that the line L_1L_2 of $\langle S_1^y,S_2^y\rangle$ contains yz. Thus, we can find $u_i \in L_i\setminus\{y\}$ such that $z \in u_1u_2$. Since $u_i \in S_i$ and X is a subspace, we obtain that $z \in X$. Therefore, $(xy)^{\perp} \cap S \subseteq X$. But $(xy)^{\perp} \cap S$ contains a symplectic pair, whence $S \subseteq D(X)$. This yields that $b \in D(X)$, proving that $D(X)$ is a subspace.

(2) $D(X)$ <u>is geodesically closed, satisfies</u> $(P3)_3$ <u>and has diameter 2.</u>

Let a_1,a_2 be noncollinear points of $D(X)$. We show that a_1,a_2 is a symplectic pair and that $a_1^\perp \cap a_2^\perp$ is contained in $D(X)$. This clearly suffices for the proof of (2).

The case where a symplecton $S \in \mathcal{S}[X]$ contains both a_1 and a_2 is obvious. Therefore, we assume that there is no such symplecton. Choose $S_i \in \mathcal{S}[X]$ such that $a_i \in S_i$ $(i=1,2)$ and set $M = S_1 \cap S_2$. Then, as before, M is a singular subspace of rank 3 on x, and $a_i^\perp \cap S_{i+1}$ are singular subspaces containing $a_i^\perp \cap M$ for all i $(i=1,2$; indices modulo 2). But $\mathrm{rk}(a_i^\perp \cap M) = 2$, so $\mathrm{rk}(a_i^\perp \cap S_{i+1}) = 3$ in view of axiom (P4) or (F4) (whichever prevails), whence $\mathrm{rk}(a_i^\perp \cap a_{i+1}^\perp \cap S_{i+1}) = 2$. In particular, a_1,a_2 is a symplectic pair.

It remains to show that $a_1^\perp \cap a_2^\perp$ is contained in $D(X)$. Suppose $c \in a_1^\perp \cap a_2^\perp$. By the assumption that there is no symplecton in $\mathcal{S}[X]$ containing both a_1 and a_2, we have that at least one of a_1,a_2, say a_1, is not collinear with x. Thus $S_1 = S(a_1,x)$. Of course, we may (and shall) restrict attention to the case where $c \notin S_1 \cap S_2$. In particular, we have $c \notin x^\perp$ (for else $c \in a_1^\perp \cap x^\perp \subseteq S$, which has just been excluded). Now consider $c^\perp \cap S_i$ for $i = 1,2$. Since $\mathrm{rk}(a_1^\perp \cap a_2^\perp \cap M) \geq 1$ we derive that $\mathrm{rk}(\{c,a_1,a_2\}^\perp \cap M) \geq 0$ from the polar space axiom applied to $a_1^\perp \cap a_2^\perp$. Take $y \in \{c,a_1,a_2\}^\perp \cap M$. Now ya_i is a line of $c^\perp \cap S_i$ $(i=1,2)$. If (P4) holds, this implies $\mathrm{rk}(c^\perp \cap S_i) = 3$. But otherwise, $P^y \cong A_{5,3}(K)$ so that $\mathrm{rk}(c^\perp \cap S_i) = 3$, too, from Proposition 5(v). Hence $x^\perp \cap c^\perp \cap S_i$ is a singular subspace of rank 2 for $i = 1,2$. It is immediate that c,x is a symplectic pair. Setting $S = S(x,c)$, we have $S \cap S_i \supseteq \langle x, x^\perp \cap c^\perp \cap S_i\rangle$, so that $\mathrm{rk}\,(S \cap S_i) = 3$.

If $M \subseteq S$, then $c^\perp \cap S_i = \langle c^\perp \cap M, a_i\rangle$ and $c^\perp \cap M = a_i \cap M$ $(i=1,2)$ by Proposition 1(ii) and consideration of ranks, so that $a_1^\perp \cap a_2^\perp$ would contain $\langle c, c^\perp \cap M\rangle$, a singular subspace of rank 3, which is absurd. Thus $S \cap S_i \neq M$ for each i $(i=1,2)$. Now, consider P^y. Since S^y, S_1^y, S_2^y are three symplecta, mutually intersecting in distinct planes, we have $\langle S^y, S_1^y, S_2^y\rangle = \langle S_1^y, S_2^y\rangle \cong A_{4,2}(K)$ by use of Proposition 3(vii) and 5(vi) applied to P^y. From this we can derive by the same argument as in step (1) that S is contained in $D(X)$. Hence $c \in D(X)$.

This shows that $a_1^\perp \cap a_2^\perp$ is contained in $D(X)$, as wanted.

(3) $D(X) \cong D_{5,5}(K)$.

For $u \in D(X)$, set $X_u = u^\perp \cap D(X)$. This is a subspace since $D(X)$ is a subspace (see(1)).

Suppose $y \in X \setminus \{x\}$. There are distinct S_1, $S_2 \in \mathcal{S}(D(X))_{xy}$ with

$rk(S_1 \cap S_2) = 3$ (by consideration of X^x). Due to Proposition 3(iv) and 5(iii) applied to P^y, the two symplecta S_1^y, S_2^y of P^y generate a subspace of P^y isomorphic to $A_{4,2}(K)$. On the other hand, they are contained in $D(X)^y$. Since $(X_y)^y = D(X)^y$ is a geodesically closed subspace of P^y of singular rank 3 (recall that maximal singular spaces in $D(X)$ on xy have rank 2 and 3), this implies $(X_y)^y \cong A_{4,2}(K)$ and $(X_y)^y = \langle S_1^y, S_2^y \rangle$. It follows from the geodesic closure of $D(X)$ that

$$D(X_y) = \bigcup_{S \in \mathcal{S}[X_y]} S \subseteq D(X).$$

But $$((X_y)_x)^x \cong (X_y)^y \cong A_{4,2}(K)$$

and $$(X_y)_x = x^{\perp} \cap D(X_y) \subseteq x^{\perp} \cap D(X) = X,$$

so that $(X_y)_x = X$, and $D(X) = D((X_y)_x) \subseteq D(X_y)$.
Hence $D(X) = D(X_y)$. Since $D(X)$ is connected, we obtain that $(X_z)^z = D(X)^z \cong$ $\cong A_{4,2}(K)$ for all $z \in D(X)$. Thus, $D(X)$ is a parapolar space which is locally $A_{4,2}(K)$. Hence, it satisfies $(F4)_{\emptyset}$. From the previous lemma, the conclusion is that $D(X) \cong D_{5,5}(K)$. □

THEOREM 4. *Let* (P,L) *be a parapolar space of finite singular rank satisfying* $(P3)_3$ *and* $(F4)_{\{-1,0\}}$. *Then one of the following holds for some field* K.

(i) (P,L) *is a polar space of rank* 4.

(ii) $(P,L) \cong D_{5,5}(K)$

(iii) $(P,L) \cong D_{6,6}(K)$.

PROOF. Let $x \in P$. In view of Lemma 3, P^x satisfies $(P3)_2$ and $(F4)_{\{-1\}}$ and has diameter 2. Thus by [8], and finiteness of its singular rank, P^x is either a polar space or of type $A_{s,2}$ for some $s \in \mathbb{N}$, $s \geq 3$. If P^x is a polar space, then (P,L) is a polar space of rank 4 according to Lemma 2. Thus we may assume that P^x is of type $A_{n,2}$ for some $n \geq 4$. In light of Lemma 5 there is a field K such that $S \cong D_{4,1}(K)$ for any symplecton S of (P,L) and such that $P^y \cong A_{s,2}(K)$ for any $y \in P$. Applying the above proposition, we obtain a family $\mathcal{D}$ of geodesically closed subspaces of (P,L) isomorphic to

$D_{5,5}(K)$ such that for any pair x,X of a point x and a subspace X with $x \in X \subseteq x^{\perp}$ and $X^x \cong A_{4,2}(K)$ there is a unique member $D(X)$ of $\mathcal{D}$ containing X. Let $D \in \mathcal{D}$. If $D = P$, then (P,L) is of type $D_{5,5}$, and assertion (ii) holds. We therefore remain with the case where $D \neq P$ and, hence, $s \geq 5$. Take $z \in P\backslash D$.

First of all, we claim that $(F4)_{\emptyset}$ does not hold. For otherwise $rk(z^{\perp} \cap S) = 3$ for each symplecton $S \in \mathcal{S}(D)$, implying that $rk((x^y)^{\perp} \cap S^y) = 2$ for each $y \in D$ and each symplecton $S \in \mathcal{S}(D)_y$, which is absurd in view of $D^y \cong A_{4,2}(K)$ and $P^y \cong A_{s,2}(K)$ (cf. Proposition 3(viii)).

Thus, with regard to $(F4)_{\{0\}}$, there is a point x in D and a symplecton $S \in \mathcal{S}(D)_x$ such that $z^{\perp} \cap S = \{x\}$. Note that $z^{\perp} \cap D$ is a clique as D is geodesically closed. In view of $(F4)_{\{0\}}$ applied to z we see that $z^{\perp} \cap D \cap T \neq \emptyset$ for any $T \in \mathcal{S}(D)$. By the structure of $D(\cong D_{5,5}(K))$ we therefore have $rk(z^{\perp} \cap D) = 4$ (cf. Proposition 6(xiii)). Now consider P^x. For any $y \in x^{\perp}\backslash D$ the singular subspace $(x^y)^{\perp} \cap D^y$ of P^y has rank 3. But $P^x \cong A_{s,2}(K)$ and $D^x \cong A_{4,2}(K)$, so $s = 5$. In particular, maximal cliques have rank 3 and 5.

We construct a geometry of type D_6 as follows. Set $\Gamma_1 = \mathcal{D}$, $\Gamma_2 = \mathcal{S}$, $\Gamma_3 = M^{(3)}$, $\Gamma_4 = L$, $\Gamma_5 = M^{(5)}$, $\Gamma_6 = P$ and $\Gamma = \bigcup_{1\leq i\leq 6} \Gamma_i$. Define incidence $*$ on Γ by $\gamma_i * \gamma_j$ for $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$ $(1\leq i,j\leq 6)$ to be symmetrized containment (i.e., $\gamma_i \subseteq \gamma_j$ or $\gamma_j \subseteq \gamma_i$) whenever $\{i,j\} \neq \{1,5\}$, $\{2,5\}$, $\{3,5\}$ and $rk(\gamma_i \cap \gamma_j) = |j-i|$ otherwise. Then $(\Gamma,*)$ is a 6-partite looped graph which is easily seen to be a geometry of type D_6. Similar to the proof of Lemma 6, the axioms (LL) and (O) of Section 2.6 are easily verified. From Theorem 3, it now follows that Γ is a building of type D_6, and in fact (cf. [12]) the unique thick building $D_6(K)$ up to isomorphy, so that $(P,L) \cong D_{6,6}(K)$. □

4.3. PROOF OF THEOREM 1. Recall that the 'only if' part is dealt with by Proposition 2.

(i) Let $k \geq 2$ and suppose (P,L) is a parapolar space of finite singular rank s. If (P,L) is a polar space, it has rank $s+1$ by Lemma 3. Assume from now on that (P,L) is not a polar space.

(ii) If $k = 2$, then (P,L) is as described in statement (ii) due to [8].

(iii) Let $k = 3$. Then by Lemma 3 and Proposition 8 there is a field K such that (P,L) is locally $A_{s,2}(K)$, while $s \geq 4$, and there is a collection

$\mathcal{D}$ of geodesically closed subspaces of $(P,\mathcal{L})$ isomorphic to $D_{5,5}(K)$ such that for any pair x,X of a point x of P and a subspace X of P with $x \in X \subseteq x^{\perp}$ and $X^x \cong A_{4,2}(K)$. There is a unique member $D(X)$ of $\mathcal{D}$ containing X. Let $x,y,z \in P$ with $d(x,y) = 2$, $d(y,z) = 1$ and $\{x,y,z\}^{\perp}$ a maximal clique in $\{x,y\}^{\perp}$. Then from $P^y \cong A_{s,2}(K)$ and $rk(z^{\perp} \cap S(x,y)) = 3$ it follows that $S(x,y)^y$ and $(yz)^y$ generate a subspace, say Y, of P^y isomorphic to $A_{4,2}(K)$. Thus, if X is a subspace of P such that $y \in X \subseteq y^{\perp}$ and $X^y = Y$, we have $x,y,z \in D(X) \in \mathcal{D}$. On the other hand, any member of $\mathcal{D}_{\{x,y,z\}}$, must contain X such that $D(X)$ is the unique member of $\mathcal{D}_{\{x,y,z\}}$. This shows that (iii) holds if $k = 3$.

(iv) Let $k = 4$. Take $x \in P$ and consider P^x. By Lemma 3 it is a parapolar space of diameter 2 and of finite singular rank satisfying $(P3)_3$ and $(F4)_{\{-1\}}$, which is not a polar space. Application of Lemma 6 yields that $P^x \cong D_{5,5}(K)$ for some field K. Due to Lemma 5 any maximal singular subspace is isomorphic to either $A_{5,1}(K)$ or $A_{4,1}(K)$ and any symplecton is isomorphic to $D_{5,1}(K)$. Now consider the 6-partite looped graph $(\Gamma,*)$ on $\Gamma = \bigcup_{1\le i\le 6} \Gamma_i$, where $\Gamma_1 = P$, $\Gamma_2 = \mathcal{L}$, $\Gamma_3 = \mathcal{V}$, $\Gamma_4 = \mathcal{M}^{(5)}$, $\Gamma_5 = \mathcal{M}^{(4)}$ and $\Gamma_6 = \mathcal{S}$, in which incidence $\gamma_i * \gamma_j$ for $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$ is defined by symmetrized containment if $\{i,j\} \neq \{4,5\}, \{4,6\}$ and by $rk(\gamma_i \cap \gamma_j) = 2 + |i-j|$ otherwise. Then Γ is a geometry of type E_6. Moreover, (LL) holds as $(P,\mathcal{L})$ is a linear space, and $(0)_i$ for $i = 2,3$ is trivially satisfied by the construction of Γ. From Theorem 3 we obtain that Γ is a building of type E_6. If readily follows that Γ is the thick building $E_6(K)$, up to isomorphism, so that $(P,\mathcal{L}) \cong E_{6,1}(K)$.

(v) $k = 5$. Take $x \in P$. According to Lemma 3, P^x is a parapolar space of diameter 2 of finite singular rank satisfying $(P3)_4$ and $(F4)_{\{-1\}}$, but not a polar space. Thus by the previous case, there is a field K such that $P^x \cong E_{6,1}(K)$. Let $(\Gamma,*)$ be the 7-partite looped graph on $\Gamma = \bigcup_{1\le i\le 7} \Gamma_i$, where $\Gamma_1 = P$, $\Gamma_2 = \mathcal{L}$, $\Gamma_3 = \mathcal{V}$, $\Gamma_4 = \mathcal{V}^{(3)}$, $\Gamma_5 = \mathcal{M}^{(6)}$, $\Gamma_6 = \mathcal{M}^{(5)}$, $\Gamma_7 = \mathcal{S}$, in which $\gamma_i * \gamma_j$ for $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$ is defined by symmetrized containment if $\{i,j\} \neq \{5,6\}, \{5,7\}$ and by $rk(\gamma_i \cap \gamma_j) = 3 + |i-j|$ otherwise. Then Γ is a geometry of type E_7 (note that symplecta are isomorphic to $D_{6,1}(K)$ and maximal singular subspaces have rank either 5 or 6 according to Lemma 5). Now (LL) and $(0)_i$ for $i = 2,3,4$ are verified similarly to the previous case, while (LH) follows as symplecta are subspaces. Thus, we derive in the same manner as above, that $\Gamma \cong E_7(K)$ and $(P,L) \cong E_{7,1}(K)$.

(vi) $k \geq 6$. Let $k = 6$. Reasoning as before, we obtain that the residue of any point is a parapolar space of finite singular rank satisfying $(P3)_5$ and $(F4)_{\{-1\}}$ and of diameter 2, but not a polar space. Thus its residue must be isomorphic to $E_{7,1}(K)$ for some field K, an incidence system of diameter 3 (cf. Proposition 2). This absurdity shows that each parapolar space satisfying $(P3)_k$ for $k = 6$ and $(F4)_{\{-1,0\}}$ must be a polar space. By induction on k, we obtain the same result for all $k \geq 6$. This ends the proof of Theorem 1. □

4.4. PROPOSITION 9. *Suppose* K *is a field and* (P,L) *is a parapolar space which is locally* $D_{6,6}(K)$. *Then there is a collection* $\mathcal{E}$ *of geodesically closed subspaces isomorphic to* $E_{6,1}(K)$ *such that for any pair* x,X *of a point* x *and a subspace* X *of* P *with* $x \in X \subseteq x^{\perp}$ *and* $X^x \cong D_{5,5}(K)$, *there is a unique member* E(X) *of* $\mathcal{E}$ *containing* X.

PROOF. Since $S^x \cong D_{4,1}(K)$ for any $x \in P$ and $S \in \mathcal{S}_x$, we get that any symplecton is of type $D_{5,1}$. Thus (P,L) satisfies $(F3)_4$ and $(F4)_{\{-1,1\}}$ (cf. Proposition 7). Analogously to the proof of Proposition 8, the following notions are introduced for a point x and a subspace X of P with $x \in X \subseteq x^{\perp}$ and $X^x \cong D_{5,5}(K)$:

$$\mathcal{S}[X] = \{S(y,z) \mid y \in X\backslash\{x\},\ z \in X\backslash y^{\perp}\}.$$

$$E(X) = \bigcup_{S \in \mathcal{S}[X]} S.$$

Again, $\mathcal{S}[X]$ is well defined as any noncollinear pair of points in $D_{5,5}(K)$ is symplectic. Clearly, $x^{\perp} \cap E(X) = X$. For any $u \in E(X)$, set $X_u = u^{\perp} \cap E(X)$.

Suppose $y \in X\backslash\{x\}$. Then $y^{\perp} \cap z^{\perp} \subseteq y^{\perp} \cap E(X)$ for any $z \in X\backslash y^{\perp}$. On the other hand, if $u \in y^{\perp} \cap S$ for some $S \in \mathcal{S}[X]$, then either $y \in S$ or $y^{\perp} \cap S$ is a maximal singular subspace of S. In both cases there is $z \in u^{\perp} \cap S\backslash y^{\perp}$ with $u \in y^{\perp} \cap z^{\perp}$. We have shown,

(*) $$X_y = \bigcup_{z\in X\setminus y^\perp} S(x,y) = \bigcup_{z\in X\setminus y^\perp} x^\perp \cap y^\perp.$$

Again, we proceed in three steps.

(1) $E(X)$ *is a subspace of* (P,L).

Let a_1,a_2 be distinct collinear points of $E(X)$, and take $b \in a_1a_2\setminus\{a_1,a_2\}$. If a symplecton from $S[X]$ contains both a_1 and a_2, there is nothing to prove. Thus we may, and shall, restrict to the case where $a_1a_2 \cap x^\perp = \emptyset$. Choose $S_i \in S[X]$ such that $a_i \in S_i$ $(i=1,2)$ and set $M = S_1 \cap S_2$. Then M is either a singular subspace of rank 4 on x or $M = \{x\}$ by consideration of the residue $X^x \cong D_{5,5}(K)$ on x (cf. Proposition 6 (iii)). But $a_2 \in a_1^\perp \cap S_2$, so $a_1^\perp \cap S_2$ contains a line L on a_2 in view of axiom $(F4)_{\{-1,1\}}$. As x, L are both in S_2, there is a point $z \in x^\perp \cap L$. Now $z \in x^\perp \cap a_1^\perp \cap a_2^\perp \subseteq S_1 \cap S_2\setminus\{x\}$, so that M has rank 4. Since $a_i^\perp \cap S_j$ is a clique containing a_j when $\{i,j\} = \{1,2\}$ we have $a_1^\perp \cap S_1 \cap S_2 =$ $= a_1^\perp \cap a_2^\perp \cap S_1^\perp \cap S_2 = a_2^\perp \cap S_1 \cap S_2$ so that $a_1^\perp \cap M = a_2^\perp \cap M$, a singular subspace of rank 3.

Let $b \in a_1a_2$. Then $x^\perp \cap b^\perp$ contains $b^\perp \cap M = a_1^\perp \cap M$, so x,b is a symplectic pair. Set $S = S(x,b)$. Observe that $M = \langle b^\perp\cap M,x\rangle \subseteq S$, so that $M = S \cap S_1 \cap S_2$. Let Y be a line of $a_1^\perp \cap M$. Reasoning as in the proof of Proposition 8, we obtain that the residue of S at Y (i.e. the residue of S^y at Y within the residue P^y at a point y of Y) is contained in the subspace of the residue at Y generated by the residues of S_1,S_2 at Y. Continuing in the analogue of the proof of Proposition 8 (but with Y substituted for y), we obtain $\langle x,Y\rangle^\perp \cap S \subseteq X$, and $S \subseteq E(X)$. Thus $b \in E(X)$, proving (1).

(2) $E(X)$ *is geodesically closed, satisfies* $(P3)_4$ *and has diameter* 2.

Let a_1,a_2 be points of $E(X)$ with $a_1 \notin a_2^\perp$. We show that a_1,a_2 is a symplectic pair and that $a_1^\perp \cap a_2^\perp \subseteq E(X)$. This clearly suffices for the proof of (2). The case where a symplecton from $S[X]$ contains both a_1 and a_2 being obvious, we may and shall assume that there is no such symplecton.

For $i = 1,2$, choose $S_i \in S[X]$ such that $a_i \in S_i$ $(i=1,2)$. Since S_1^x,S_2^x are symplecta in $X^x \cong D_{6,6}(K)$, we have either $rk(S_1\cap S_2) = 4$ or $S_1 \cap S_2 = \{x\}$ (cf. Proposition 6 (iii)). Note that a_1,a_2 are not both in X, for otherwise $S(a_1,a_2)$ would be a symplecton in $S[X]$ containing a_1,a_2. Without loss of generality, we have $a_1 \notin x^\perp$.

First, suppose $\mathrm{rk}(S_1 \cap S_2) = 4$. Then $a_1^\perp \cap a_2^\perp \cap S_1 \cap S_2$ is a singular subspace of $a_1^\perp \cap a_2^\perp$ of rank ≥ 2. It follows that a_1, a_2 is a symplectic pair. Let $c \in a_1^\perp \cap a_2^\perp$. Note that $c^\perp \cap a_1^\perp \cap a_2^\perp \cap S_1 \cap S_2$ contains a line, say, L (since both c and $a_1^\perp \cap a_2^\perp \cap S_1 \cap S_2$ are in the polar space $a_1^\perp \cap a_2^\perp$). If $c \in S_1 \cup S_2$, there is nothing to prove. Assume, therfore, that $c \notin S_1 \cup S_2$. Then, in particular $c \notin x^\perp$ (for else $c \in a_1^\perp \cap x_1^\perp \subseteq S(a_1,x) = S_1$). But $L \subseteq x^\perp \cap c^\perp$ so x,c is a symplectic pair. Set $S = S(x,c)$. Taking the residue at L and using the same arguments as in step (2) of the proof of Proposition 8, we obtain that S is contained in $E(X)$, whence $c \in E(X)$.

Now, suppose $S_1 \cap S_2 = \{x\}$. If $a_2 \in X$, then $a_2^\perp \cap S_1$ contains x, and hence by axiom $(F4)_{\{-1,1\}}$, a line U. Taking $w \in S_1 \cap U^\perp \backslash a_2^\perp$, we obtain that w,a_2 is a symplectic pair and $S_1 \cap S(w,a_2) \supseteq U$. Since both S_1 and $S(w,a_2)$ are members of $\mathcal{S}[X]$, this yields that $S_1 \cap S(w,a_2)$ is a singular subspace of rank 4. Replacing S_2 by $S(w,a_2)$, we have $\mathrm{rk}(S_1 \cap S_2) = 4$ again.

Therefore, it remains to consider the case where $a_2 \notin X$. We first show that a_1,a_2 is a symplectic pair. Take $y_1 \in x^\perp \cap a_1^\perp$ and consider $y_1^\perp \cap S_2$. In view of $X^x \cong D_{5,5}(K)$ and Proposition 6, it follows from $y_1 \in X$ and $S_2 \in \mathcal{S}[X]$ that $\mathrm{rk}(y_1^\perp \cap S_2) = 4$. Thus $\mathrm{rk}(a_2^\perp \cap y_1^\perp \cap S_2) \geq 3$, and in fact equality holds, as $x \in y_1^\perp \cap S_2 \backslash a_2^\perp$. In particular, y_1,a_2 is a symplectic pair. But $a_1^\perp \cap S(y_1,a_2)$ contains y_1 and hence a line, due to $(F4)_{\{-1,1\}}$. Consequently, we can find $z_1 \in a_2^\perp \cap a_1^\perp \cap S(y_1,a_2) = a_1^\perp \cap a_2^\perp \cap y_1^\perp$. In particular, $d(a_1,a_2) = 2$. The same argument, but now with indices 1 and 2 interchanged, leads to a point $z_2 \in a_1^\perp \cap a_2^\perp \cap y_2^\perp$ for any $y_2 \in x^\perp \cap a_1^\perp$. Suppose that a_1,a_2 is a special pair, i.e., $a_1^\perp \cap a_2^\perp = \{z\}$ for some $z \in P$.
Then $z = z_1 = z_2 \in S(y_1,a_2) \cap S(y_2,a_1) \subseteq y_1^\perp \cap y_2^\perp$, so $z \in y_1^\perp \cap y_2^\perp$. Moreover, for any $y \in x^\perp \cap a_1^\perp$, we have $z \in y^\perp$ by the above argument for y_1 instead of y. Obviously this means $z \in S_1$. Similarly, we have $z \in S_2$, so that $z \in S_1 \cap S_2 = \{x\}$, or $z = x$, which is absurd as $x \notin a_1^\perp \cap a_2^\perp$. We conclude that a_1,a_2 is a symplectic pair.

Next let $c \in a_1^\perp \cap a_2^\perp$. If $c \in S_1 \cup S_2$, then $c \in E(X)$. Suppose $c \notin S_1 \cup S_2$. Let $i = 1,2$. In view of axiom $(F4)_{\{-1,1\}}$, we get from $a_i \in c^\perp \cap S_i$ that $c^\perp \cap S_i$ contains a line on a_i. Let u_i be the unique point on this line collinear with x. Since $S_1 \cap S_2 = \{x\}$ and $x \notin a_i^\perp$, we have $u_1 \neq u_2$. But $u_1,u_2 \in c^\perp \cap x^\perp$, so c,x is a symplectic pair set $S = S(x,c)$. Note that

$S \neq S_i$ since $c \notin S_1 \cup S_2$. Now, in the residue $P^{u_2} \cong D_{6,6}(K)$ at u_2, the two distinct symplecta S^{u_2} and $S_2^{u_2}$ have the point xu_2 in commen, whereas $L_{u_2}(<u_2,a_2,c>)$ is a line of P^{u_2} having points cu_2 and a_2u_2 with distance 2 to xu_2. By Proposition 7 this yields that $rk(S^{u_2} \cap S_i^{u_2}) = 3$ so that $rk(S \cap S_i) = 4$. Consequently, $rk(c^\perp \cap S \cap S_i) = 3$, so that $rk(c^\perp \cap S_i) = 4$ according to axiom $(F4)_{\{-1,1\}}$. Moreover, $x \notin c^\perp$, as $c \in x^\perp$ would imply $c \in a_1^\perp \cap a_2^\perp \cap x^\perp \subseteq S_1 \cap S_2$. Therefore, we have $rk(x^\perp \cap c^\perp \cap S_i) = 3$. We assert that there are $w_i \in x^\perp \cap c^\perp \cap S_i$ $(i=1,2)$ such that $w_1 \notin w_2^\perp$. For otherwise $x^\perp \cap c^\perp \cap (S_1 \cup S_2)$ would be a clique consisting of two disjoint singular subspaces of rank 3, so that $<x^\perp \cap c^\perp \cap (S_1 \cup S_2)>$ would be a singular subspace of rank 7, contradicting the fact that the rank of a maximal singular subspace is either 4 or 6.

Finally, taking $w_i \in x^\perp \cap c^\perp \cap S_i$ $(i=1,2)$ with $w_1 \notin w_2^\perp$, we obtain that w_1,w_2 is a symplectic pair in X, so that $c \in S(w_1,w_2) \subseteq E(X)$. We conclude that $a_1^\perp \cap a_2^\perp \subseteq E(X)$. This establishes (2).

(3) $E(X) \cong E_{6,1}(K)$

Let $y \in X\backslash\{x\}$ and observe that $X_y = y^\perp \cap E(X)$ since both $y^\perp$ and $E(X)$ are subspaces. We claim that the residue $(X_y)^y$ of X_y at y is isomorphic to $D_{5,5}(K)$. For, there are distinct $S_1,S_2 \in \mathcal{S}(E(X))_{xy}$ with $rk(S_1 \cap S_2) = 4$ (recall that $X^x \cong D_{5,5}(K)$). Since $P^y \cong D_{6,6}(K)$, the subspace $<S_1^y,S_2^y>$ of P^y is a geodesically closed subspace of $E(X)^y$ isomorphic to $D_{5,5}(K)$, (cf. Proposition 6). Since any subspace of $D_{6,6}(K)$ isomorphic to $D_{5,5}(K)$ is a maximal geodesically closed subspace, this yields that either $E(X)^y \cong D_{5,5}(K)$ or $E(X)^y = P^y$. However in the latter case we would have $x^\perp \cap y^\perp = x^\perp \cap y^\perp \cap E(X) \subseteq x^\perp \cap E(X) = X$, so that xy would be a point of X^x with residue entirely contained in X^x. By Lemma 4, this would imply that $X^x = P^x$, which is absurd.

So far, we have that $E(X)^y \cong D_{5,5}(K) \cong E(X)^x = X^x$. On the other hand, from (2) we obtain.

$$E(X_y) = \bigcup_{S \in \mathcal{S}[X_y]} S \subseteq E(X).$$

By an argument completely analogous to the one in step (3) of the proof of Proposition 8, the converse inclusion, and hence $E(X) = E(X_y)$ can be derived. Since $E(X)$ is connected, it follows that $(X_z)^z = E(X)^z \cong D_{5,5}(K)$ for all $z \in E(X)$. Thus $E(X)$ is a parapolar space of singular rank 5 which is locally $D_{5,5}(K)$. Hence it satisfies $(F4)_{\{-1\}}$. From Theorem 1 we conclude that $E(X) \cong E_{6,1}(K)$. $\square$

4.5. PROOF OF THEOREM 2. In view of Proposition 2 we need only deal with the 'if' part.

(i) If for any $x \in P$ the residue P^x is a polar space, then we are in case (i) by Proposition 1. Hence we may and shall, assume that the residue of no point of (P,L) is a polar space. Note that the residue P^x of $x \in P$ satisfies $(F3)_{k-1}$ and $(F4)_{\{0\}}$.

(ii) Let $k = 3$. By Lemma 5 (iii) and Proposition 3 (iii) there is a field K such that (P,L) is either locally $A_{4,2}(K)$ or locally $A_{5,3}(K)$. In the first case, $(F4)_{\{-1\}}$ holds, so we get from Theorem 1 and Proposition 7 (iii) that $(P,L) \approx D_{5,5}(K)$ for some field K. Thus we remain with the case where (P,L) is locally $A_{5,3}(K)$. There is a collection $\mathcal{D}$ of geodesically closed subspaces isomorphic to $D_{5,5}(K)$ as described in Proposition 6. Moreover any symplecton is isomorphic to $D_{4,1}(K)$.

Let $\approx$ be the relation on $\mathcal{D}$ defined by $D_1 \approx D_2$ if and only if $D_1 \cap D_2 \in M^{(4)}$ for $D_1,D_2 \in \mathcal{D}$. It is our intention to show that the graph $(\mathcal{D},\approx)$ has precisely two connected components. For $D_1,D_2 \in \mathcal{D}$, let $d_{\approx}(D_1,D_2)$ denote the distance between D_1 and D_2 within this graph, and set $D_1^T = \{D \in \mathcal{D} \mid d_{\approx}(D_1,D) \le 1\}$.

(1) *If* $x \in P$ *and* $D_1,D_2 \in \mathcal{D}_x$, *then either* $D_1 \in D_2^T$ *or* $D_1 \cap D_2 \in S$ *or* $D_1 \cap D_2 = \{x\}$.

PROOF. Recall that $D_i^x \cong A_{4,2}(K)$ while $P^x \cong A_{5,3}(K)$. Thus in the residue P^x we have either $D_1^x = D_2^x$ or $D_1^x \cap D_2^x \in (M^{(4)})^x \cup S^x$ or $D_1^x \cap D_2^x = \emptyset$ (cf. Proposition 5 (vii)). Since D_1,D_2 are connected and geodesically closed, so is $D_1 \cap D_2$. As a symplecton is a maximal geodesically closed subspace of a member of $\mathcal{D}$ (cf. Proposition 3 (iv)), and a maximal singular subspace is a maximal geodesically closed subspace of a symplecton, it follows that if $D_1 \neq D_2$ we have either $D_1 \cap D_2 \in M^{(4)} \cup S$ or $D_1 \cap D_2 = \{x\}$.

(2) *Suppose* D_1,D_2,D_3 *are distinct members of* $\mathcal{D}$ *satisfying* $D_1 \approx D_2 \approx D_3$. *Then*

the following three statements are equivalent:

(a) $D_1 \cap D_3 \neq \emptyset$.

(b) $D_1 \cap D_2 \cap D_3 \neq \emptyset$.

(c) $D_1 \approx D_3$.

Moreover, if these statements hold, then $D_1 \cap D_2 \cap D_3 \in \mathcal{M} \cup \mathcal{L}$.

PROOF. The implication "(c) ⇒ (a)" being trivial, we shall only treat "(a) ⇒ (b)" and "(b) ⇒ (c)". Set $M_i = D_i \cap D_{i+1}$ for $i = 1,2,3$ (indices modulo 3).

"(a) ⇒ (b)". Suppose that $M_3 \neq \emptyset$ and $D_1 \cap D_2 \cap D_3 = \emptyset$. Fix $x \in M_3$. From the structure of $D_i \cong D_{5,5}(K)$ we have $rk(x^\perp \cap M_i) = -1,0,2$ for each $i = 1,2$.

First of all, suppose that $rk(x^\perp \cap M_1) = rk(x^\perp \cap M_2) = 2$. Note that $M_1 \cap M_2 = \emptyset$ as $D_1 \cap D_2 \cap D_3 = \emptyset$. If $x^\perp \cap (M_1 \cup M_2)$ were a clique then $rk\langle x^\perp \cap M_1, x^\perp \cap M_2\rangle \geq 5$, which conflicts with the singular rank of D_2. Hence there are $x_i \in x^\perp \cap M_i$ $(i=1,2)$ with $x_1 \notin x_2^\perp$, so that $x \in x_1^\perp \cap x_2^\perp \subseteq D_2$ by geodesic closure of D_2. But then $x \in D_1 \cap D_2 \cap D_3$, contradiction. Thus $rk(x^\perp \cap M_i) \leq 0$ for at least one $i \in \{1,2\}$, say $i = 2$.

Suppose $rk(x^\perp \cap M_1) = 2$. Since $\{z \in M_1 \mid z^\perp \cap M_2 \neq \emptyset\}$ is subspace of M_1 of rank 3, it follows from Proposition 6, that there is $z \in x^\perp \cap M_1$ with $z^\perp \cap M_2 \neq \emptyset$. Now $rk(z^\perp \cap M_2) = 2$ (cf. Proposition 6), so there exists $y \in z^\perp \cap M_2 \backslash x^\perp$. Then $x,y \in D_3$ so x,y is a symplectic pair and $z \in x^\perp \cap y^\perp \subseteq S(x,y) \subseteq D_3$. Hence $z \in D_1 \cap D_2 \cap D_3$, which is absurd. Thus $rk(x^\perp \cap M_i) \leq 0$ for both $i = 1,2$. Take $y_i \in M_i$ $(i=1,2)$ with $y_2 \in y_1^\perp \backslash x^\perp$ (refer to Proposition 6 to ensure existence). Now x,y_2 is a symplectic pair as $x,y_2 \in D_3$. Observe that $y_1 \notin S(y_2,x)$ as $D_1 \cap D_2 \cap D_3 = \emptyset$. But $y_2 \in y_1^\perp \cap S(y_2,x)$, so by axiom $(F4)_{\{-1,1\}}$ there is a line on y_2 in $y_1^\perp \cap S(y_2,x)$. Let u be the point on this line collinear to x. The $u \in x^\perp \cap y_1^\perp \cap y_2^\perp \subseteq D_1 \cap D_3$ and $y_i \in u^\perp \cap M_i$, $(i=1,2)$. Thus $u \in D_1 \cap D_3$ and $rk(u^\perp \cap M_i) = 2$ for each $i = 1,2$ (see Proposition 6 and use $D_1 \cap D_3 \in M^{(4)}$). Since this possibility has been excluded above, we have the final contradiction, proving that $D_1 \cap D_2 \neq \emptyset$ implies $D_1 \cap D_2 \cap D_3 \neq \emptyset$.

"(b) ⇒ (c)". Assume $x \in D_1 \cap D_2 \cap D_3$, and consider the residue at x. Since $D_i^x \cong A_{4,2}(K)$, $M_1^x, M_2^x \cong A_{3,1}(K)$ and $P^x \cong A_{5,3}(K)$, we see from the structure of $A_{5,3}(K)$ (see Proposition 5(x)) that M_3^x is a singular subspace of P^x of rank 3. Thus M_3 contains a member of $\mathcal{M}^{(4)}$. In particular M_3 cannot be a symplecton, and $M_3 \in \mathcal{M}$. This proves $D_1 \approx D_3$.

It remains to show the last statement of step (2). Suppose that $D_1 \cap D_3 \neq \emptyset$. Then as we have seen above, $D_1 \cap D_2 \cap D_3$ contains a point. Take $x \in D_1 \cap D_2 \cap D_3$. In the residue at x, we have $D_i^x \cong A_{4,2}(K)$ and $M_i^x \cong A_{3,1}(K)$ for $i = 1,2,3$. This yields (see Proposition 5(ix)) that $\mathrm{rk}((D_1 \cap D_2 \cap D_3)^x)$ is either 3 or 0. Consequently, $\mathrm{rk}(D_1 \cap D_2 \cap D_3)$ is either 4 or 1, which is the desired statement.

(3) *Let* D_1, D_2, D_3 *be distinct members of* $\mathcal{D}$ *with* $D_1 \cap D_2 \cap D_3 \in \mathcal{M}$. *If* $D \in \mathcal{D}$ *satisfies* $D \approx D_1$, $D \approx D_2$ *then* $D \approx D_3$.

PROOF. Observe that $D \cap D_1 \cap D_2 \neq \emptyset$ in view of (2). But $D_1 \cap D_2 \cap D_3 = D_1 \cap D_2$. In particular, $D_1 \cap D_2 = D_1 \cap D_3$ and hence $D \cap D_1 \cap D_3 \neq \emptyset$. Now use (2) again.

(4) *Let* D_1, D *be in the same connected component of* $(\mathcal{D}, \approx)$. *Then* $d_{\approx}(D_1, D) \leq 2$.

PROOF. Obviously it suffices to show the following. If $D_i \in \mathcal{D}$ for $i = 1,2,3,4$ such that $D_1 \approx D_2 \approx D_3 \approx D_4$, then $d_{\approx}(D_1, D_4) \leq 2$. Thus, let $D_i \in \mathcal{D}$ for $i = 1,2,3,4$ satisfy $D_1 \approx D_2 \approx D_3 \approx D_4$. If $D_3 \in D_1^{\top}$ or $D_4 \in D_2^{\top}$, there is nothing to prove. Thus, by (2), we may assume $D_1 \cap D_3 = \emptyset$ and $D_2 \cap D_4 = \emptyset$. Since $\{z \in D_2 \cap D_3 \mid z^{\perp} \cap D_1 \cap D_2 \neq \emptyset\}$ and $\{z \in D_2 \cap D_3 \mid z^{\perp} \cap D_3 \cap D_4 \neq \emptyset\}$ are singular subspaces of $D_2 \cap D_3$ with rank 3 by Proposition 6, there is a plane V in $D_2 \cap D_3$ such that for any point $z \in V$, we have $\mathrm{rk}(z^{\perp} \cap D_1 \cap D_2) = \mathrm{rk}(z^{\perp} \cap D_3 \cap D_4) = 2$. Let z_1, z_2 be distinct points of V. Then $z_i^{\perp} \cap D_1 \cap D_2$ are planes in the singular subspace $\{z \in D_1 \cap D_2 \mid z^{\perp} \cap D_2 \cap D_3 \neq \emptyset\}$ of rank 3. This yields that $z_1^{\perp} \cap z_2^{\perp} \cap D_1 \cap D_2$ has rank at least 1, so that there is a line $L_1 \in \mathcal{L}(D_1 \cap D_2)$ with $L_1 \subseteq L_2^{\perp}$, where $L_2 = z_1 z_2$. Similary, we can find $L_3 \in \mathcal{L}(D_3 \cap D_4)$ with $L_3 \subseteq L_2^{\perp}$. Set $M_2 = L_1^{\perp} \cap L_2^{\perp}$ and $M_3 = L_2^{\perp} \cap L_3^{\perp}$. Then M_2 is the unique maximal singular subspace in D_2 intersecting the two maximal singular subspaces $D_1 \cap D_2$ and $D_2 \cap D_3$ in the lines L_1 and L_2 respectively. In particular, $M_2 \subseteq D_2$. Similarly, $M_3 \subseteq D_3$.

Take $x \in L_2$ and consider the residue at x. Now D_2^x, D_3^x are subspaces of P^x isomorphic to $A_{4,2}(K)$ and $(D_2 \cap D_3)^x$, M_2^x, M_3^x are subspaces isomorphic to $A_{3,1}(K)$, while $M_i^x \subseteq D_i^x$ for $i = 2,3$ and $\mathrm{rk}(M_2^x \cap M_3^x) \geq 0$. Due to the structure of $P^x \cong A_{5,3}(K)$, this yields the existence of a subspace of P^x isomorphic to $A_{4,2}(K)$ containing M_2^x and M_3^x (cf. Proposition 5 (xi)). Therefore, there is a subspace X of P with $x \in X \subseteq x^{\perp}$ and $X^x \cong A_{4,2}(K)$ such that $M_2 \cup M_3 \subseteq X$. But then $D = D(X)$, as defined in Proposition 8, is a member of $\mathcal{D}$ containing $M_2 \cup M_3$. Thus $D \cap D_i \supseteq M_i$ for $i = 2,3$, so that $D_2 \approx D \approx D_3$. But also $D_1 \cap D_2 \cap D \supseteq D_1 \cap M_2 \supseteq L_1$, whence $D_1 \approx D$ in view of (2). Similarly, $D_4 \approx D$ as $D_3 \cap D_4 \cap D \supseteq D_4 \cap M_3 \supseteq L_2$. It follows that $D_1 \approx D \approx D_4$, so that $d_{\approx}(D_1,D_4) \leq 2$. This settles (4).

Fix $D_1,D_2 \in \mathcal{D}$ *with* $D_1 \cap D_2 \in S$ *and let* $\mathcal{D}^i$ *for* $i = 1,2$ *be the connected component of* D_i *in* $(\mathcal{D},\approx)$.

By (2) and (4), we have that $\mathcal{D}^1,\mathcal{D}^2$ are disjoint connected conponents of $(\mathcal{D},\approx)$.

(5) $\mathcal{D} = \mathcal{D}^1 \,\dot{\cup}\, \mathcal{D}^2$.

<u>PROOF.</u> Take $x \in D_1 \cap D_2$, and let $D \in \mathcal{D}$. If $x \in D$, then $D \in \mathcal{D}^1 \cup \mathcal{D}^2$ follows from consideration of the residue at x. For $x \notin D$, apply induction with respect to $d(x,D)$. Let $y \in D$ and $z \in y^{\perp}$ be such that $d(x,z) = d(x,D) - 1$. Then, as in the first step, D is connected within $(\mathcal{D},\approx)$ to a member, say E, of $\mathcal{D}$ on yz. But by induction, $E \in \mathcal{D}^1 \cup \mathcal{D}^2$, and therefore $D \in \mathcal{D}^1 \cup \mathcal{D}^2$.

We now construct a geometry $(\Gamma,*)$ of type E_6 as follows. Put $\Gamma_1 = \mathcal{D}^1$, $\Gamma_2 = \bigcup_{D\in\mathcal{D}^1} M(\mathcal{D})$, $\Gamma_3 = L$, $\Gamma_4 = P$, $\Gamma_5 = \bigcup_{D\in\mathcal{D}^2} M(\mathcal{D})$, $\Gamma_6 = \mathcal{D}^2$. From (5), it is immediate that $M = \Gamma_2 \cup \Gamma_5$. Set $\Gamma = \bigcup_{1\leq i\leq 6} \Gamma_i$ and define $\gamma_i * \gamma_j$ for $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ by symmetrized containment for $\{i,j\} \neq \{1,6\}$, $\{2,6\}$, $\{1,5\}$, $\{2,5\}$ and as follows for the other cases

$$\gamma_1 * \gamma_6 \iff \gamma_1 \cap \gamma_6 \in S$$
$$\gamma_1 * \gamma_5 \iff \gamma_1 \cap \gamma_5 \in V^{(3)}$$
$$\gamma_2 * \gamma_6 \iff \gamma_2 \cap \gamma_6 \in V^{(3)}$$
$$\gamma_2 * \gamma_5 \iff \gamma_2 \cap \gamma_5 \in V\,.$$

It is straightforward to verify that Γ is a geometry of type E_6. We next verify that Γ is a building of type E_6. According to Theorem 3, it suffices to check axioms (LL) and $(0)_i$ for $i = 2,3$.

Now, (LL) states that any two maximal singular subspaces contained in two distinct members of $\mathcal{D}^1$ coincide. But this is obvious from the definition of $\mathcal{D}^1$ and (1).

In order to establish $(0)_2$ and $(0)_3$ it suffices to show, that for any $M \in \Gamma_2$ and $L \in \Gamma_3$, there exists $D \in \mathcal{D}^1$ with $M \subseteq D$ and $L \nsubseteq D$. Let $M \in \Gamma_2$ and $L \in \Gamma_3$. Considering the residue at a point of $M \cap L$ if $M \cap L \neq \emptyset$, we can easily reduce the argument to the case where $M \cap L = \emptyset$. There is at most one member of $\mathcal{D}^1$ containing $M \cup L$. On the other hand, it is obvious from consideration of the residue at a point of M, that there is more than one member of $\mathcal{D}^1$ on M. This leads to $D \in \mathcal{D}^1$ as desired.

Similarly one can show that $(0)_3$ holds for the geometry Γ. We conclude that Γ is a building of type E_6. However, (P,L) is locally isomorphic to $A_{5,3}(K)$, so Γ is the thick building $E_6(K)$. As a consequence, $(P,L) \cong E_{6,4}(K)$.

(iii) Let k = 4. By Theorem 4, we have for $x \in P$ that its residue P^x is isomorphic to either $D_{5,5}(K)$ or $D_{6,6}(K)$ for some field K. Due to an argument involving the ranks of maximal singular subspaces, this yields the existence of a field K such that (P,L) is either locally $D_{5,5}(K)$ or locally $D_{6,6}(K)$. In the former case, (P,L) actually satisfies $(F4)_{\{-1\}}$, so that $(P,L) \cong E_{6,1}(K)$ according to Theorem 1. Thus, we may assume $P^x \cong D_{6,6}(K)$ for all $x \in P$.

Now, maximal singular subspaces are isomorphic to $A_{6,1}(K)$ or $A_{4,1}(K)$, symplecta are isomorphic to $D_{5,1}(K)$, and by Proposition 9 there is a nonempty collection E of geodesically closed subspaces isomorphic to $E_{6,1}(K)$. Thus we can construct a geometry $(\Gamma,*)$ of type E_7 as follows. Put $\Gamma_1 = E$, $\Gamma_2 = S$, $\Gamma_3 = M^{(4)}$, $\Gamma_4 = V^{(2)}$, $\Gamma_5 = M^{(6)}$, $\Gamma_6 = L$, $\Gamma_7 = P$. Set $\Gamma = \bigcup_{1\le i\le 7} \Gamma_i$, and define incidence $\gamma_i * \gamma_j$ for $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$ by symmetrized containment if $\{i,j\} \neq \{1,5\}, \{2,5\}, \{3,5\}$ and by $\mathrm{rk}(\gamma_i \cap \gamma_j) = |j-i| + 1$ otherwise. Again it is straightforward to verify that Γ is a geometry of type E_7. We next check the axioms (LL), (LH) and $(0)_i$ for i = 2,3,4 of Section 2.6..

(LL) states that any two symplecta contained in two distinct members of E must coincide. This is clearly true as symplecta are maximal geodesically closed of members of E.

As for $(0)_2$, $(0)_3$, $(0)_4$, it suffices to show that for any symplecton S and any plane V there is a member E of E with $S \subseteq E$ and $V \nsubseteq E$. But this follows easily by an argument similar to that for $(0)_2$ and $(0)_3$ in (ii).

Finally, (LH) is trivialy satisfied since two members $E, E^1 \in E$ properly containing a symplecton must coincide (as we have seen before). By Theorem

3, this settles that Γ is a building of type E_7. It follows that Γ is actually isomorphic to $E_7(K)$, so that $(P,L) \cong E_{7,7}(K)$.

(iv) If $k = 5$, then by Theorem 1, (P,L) is locally $E_{6,1}(K)$, so that in fact (P,L) satisfies $(F4)_{\{-1\}}$. But then (P,L) is of type $E_{7,1}$ by Theorem 1, which is absurd since $(F4)_{\{-1,1\}}$ does not hold for such spaces.

It follows that $k \neq 5$. If $k = 6$, then again by Theorem 1, for any $x \in P$. There is a field K such that $P^x \cong E_{7,1}(K)$. Hence, the residue at any point is isomorphic to $E_{7,1}(K)$. Thus $M = M^{(7)} \cup M^{(6)}$ and symplecta are isomorphic to $D_{7,1}(K)$. We construct $(\Gamma,*)$ as follows. Put $\Gamma_1 = P$, $\Gamma_2 = L$, $\Gamma_3 = V^{(2)}$, $\Gamma_4 = V^{(3)}$, $\Gamma_5 = V^{(6)}$, $\Gamma_6 = M^{(7)}$, $\Gamma_7 = M^{(6)}$, $\Gamma_8 = S$. Set $\Gamma = \bigcup_{1\le i\le 8} \Gamma_i$, and define incidence $\gamma_i * \gamma_j$ for $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$ by containment if $\{i,j\} \neq \{6,7\}$, $\{6,8\}$ and by $rk(\gamma_i \cap \gamma_j) = 4 + |i-j|$ otherwise. Then $(\Gamma,*)$ is easily seen to be a geometry of type E_8. According to Theorem 3 and by thickness, Γ is isomorphic to a building provided axioms (LL), (LH), (HH) and $(0)_i$ hold for $i = 2,3,4,5$.

Now, $(0)_i$ $(i=2,3,4,5)$ is satisfied by construction, (LL) reflects that (P,L) is a linear incidence system, (LH) is equivalent to saying that symplecta are subspaces, and (HH) holds because of Proposition 1. The conclusion is that $\Gamma \cong E_8(K)$ and that $(P,L) \cong E_{8,1}(K)$.

(v) By Theorem 1, the residue at a point x of P must be a polar space if $k \ge 7$. Thus (P,L) is a polar space whenever $k \ge 7$. This ends the proof of Theorem 2. □

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